Convergence of a Generalized Fast Marching Method for a non-convex eikonal equation

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Abstract

We present a new Fast Marching algorithm for a non-convex eikonal equation modeling front evolutions in the normal direction. The algorithm is an extension of the Fast Marching Method since the new scheme can deal with a *time-dependent* velocity without *any restriction on its sign*. We analyze the properties of the algorithm and we prove its convergence in the class of discontinuous viscosity solutions. Finally, we present some numerical simulations of fronts propagating in $\mathbb{R}^2$.

AMS Classification: 65M06, 65M12, 49L25.
Keywords: Hamilton Jacobi equations, fast marching scheme, convergence, viscosity solutions.

1 Introduction

The goal of this paper is to propose and analyze a numerical scheme to compute the evolution of a front driven by its normal velocity $c(x,t)$ under very general assumptions on c. In particular, we will remove the usual assumption which assigns to c a constant sign during the evolution. This means that the front can oscillate and pass several times over the same points. The initial front is the boundary of an open set Ω_0, which is represented by a characteristic function $1_{\Omega_0} - 1_{\Omega_0^c}$, defined equal to 1 on Ω_0 and -1 on its complementary set. Mathematically, we are interested in the discontinuous viscosity solution $\theta(x,t)$ of the following equation

$$(1.1) \qquad \begin{cases} \theta_t(x,t) = c(x,t)|D\theta(x,t)| & \text{on} \quad \mathbb{R}^N \times (0,T) \\ \theta(\cdot,0) = 1_{\Omega_0} - 1_{\Omega_0^c}. \end{cases}$$

Here the support of the discontinuities of the function θ localizes the front we are interested in. This work is motivated by the numerical computation of dislocations dynamics where the velocity of the front can change sign (see Rodney, Le Bouar, Finel [13]).

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A very popular method to describe the evolution of a front is the Level Sets method (see the seminal paper by Osher and Sethian [12] as well as the monographies [15, 16], [9]), where the discontinuous solution θ is replaced by a continuous function, and the equation is discretized using finite difference method with a CFL condition of the type $\Delta t \|c\|_\infty \leq \Delta x$ for explicit schemes, where Δx is the space step and Δt is the time step.

Another well-known method is the Fast Marching Method (FMM) (see Sethian [17, 14]), where the unknown of the problem is the time $t(x)$ the front reaches the point x. This method works for non negative (non positive) velocities and provides a very efficient scheme which concentrates the computational effort on a neighborhood of the front. To be more precise, keeping in mind the viewpoint of discontinuous solutions, in the usual FMM we define the Accepted region (A+) as the discretization of the region $\{\theta = 1\}$ and the Narrow Band (NB-) as the discretization of the boundary $\partial\{\theta = 1\}$, which is at the discrete level contained in the region $\{\theta = -1\}$. The algorithm computes the new values only at the nodes belonging to the narrow band and accepts just one of them, the one corresponding to the minimum value (see Kim [10] for a faster implementation). In the case when c cannot change sign we have a monotone (increasing or decreasing) evolution and the front passes just one time on every point of the computational domain. The corresponding arrival time of the front is univalued so that the evolutive problem reduces to a stationary problem (the eikonal equation). Note that in this method, there is no time step, because the time is itself the unknown of the problem so that the original evolutive problem (1.1) reduces to a stationary problem as remarked in [8] and [11].

To set this paper into perspective, let us recall that the FMM was initially developed for (1.1) with time independent velocities $c(x) > 0$ (see Sethian [14] and Tsitsiklis [20] for the method previously developed on graphs). This FMM scheme has been proved to be convergent, using a relation between the FMM solution and the numerical solution to finite difference schemes for the Level Sets formulation, for which it is known that these schemes are convergent (see Cristiani and Falcone [7]). More recently, the method has been extended to more general Hamilton-Jacobi equation by Sethian and Vladimirsky [18, 19] and it has been also adapted to the case of time-dependent non-negative velocities $c(x,t) \geq 0$ by Vladimirsky [21]. However, up to our knowledge, no proof of convergence has been given for the variable sign velocity case.

The goal of this paper is to propose a Generalized Fast Marching Method (GFMM) which works for general velocities $c(x,t)$ without sign restrictions. This implies that the evolution is not necessarily monotone and that the time of arrival of the front can be multivalued. Then, in our GFMM it is natural to introduce two Accepted regions (A_+) and (A_-), and two Narrow bands (F_+) and (F_-) in order to be able to take into account the changes of sign of the velocity. The typical picture is Fig. 1. In some sense we track two fronts : one moving with positive velocity and one moving with negative velocity. A preliminary version of this new scheme has been proposed in [5], however in that first version no proof was given and some small but very important details, which make the scheme work in the general case, were missing.

Our GFMM has a great potential for several future developments. Let us only mention the application to dislocations dynamics that we will study in a future work.

We introduce in Section 2 the GFMM scheme. Let us observe that there are several subtilities, that do not appear in the usual FMM for $c(x) > 0$. These new features seem

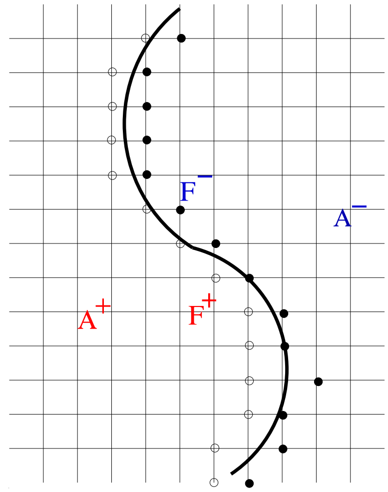

Figure 1: The narrow bands F_+ and F_-.

necessary to make the scheme work for general $c(x,t)$. Let us list a few of them. First, where the velocity changes sign in space, we need somehow to regularize it to avoid instabilities (in time) of the front. Second, because the time step is somehow the difference $\Delta t_n = t_{n+1} - t_n$ between two computed times, we need our algorithm to ensure that this time step remains bounded from above by a given time step Δt. In fact, this is necessary since the convergence result shows that to improve our approximation of the solution the discretization parameters Δt and Δx must go to 0 and when the velocity is very close to zero, if that bound is not respected, the algorithm may generate a sequence of time steps Δt_n non convergent to zero. Third, we may get computed times $\tilde{t}_{n+1}$ which satisfy $\tilde{t}_{n+1} < t_n$, if for instance the velocity is always equal to zero except at time t_n. In this case, it is necessary to update the time with the value t_n and not $\tilde{t}_n$. Fourth, when the front is close to a given point we have to choose carefully if we update or not the value of the time at this given point. This really depends on the position of the discrete front at time t_n and at time t_{n+1} and on the definition of the new accepted points.

The main result of this paper is Theorem 2.5 which shows the convergence of our GFMM algorithm. When the discontinuous solution is unique, this result states that the numerical solution converges to the discontinuous viscosity solution as $\Delta x, \Delta t$ go to zero. In the case where the discontinuous viscosity solution is not unique, the result only claims that the upper semicontinuous envelope (obtained by a $limsup^*$, see (2.8) for a definition) of the numerical solution is a discontinuous viscosity subsolution and, conversely, that the lower semicontinuous envelope (obtained by the $liminf_*$) is a supersolution.

Another novelty is the proof of convergence of this GFMM algorithm. In fact, we can not use the relation with the usual schemes for the eikonal equation as in the case of non-negative velocities $c(x) > 0$ and we need a *direct proof*. It is interesting to remark that, even in the case of non-negative velocity, our proof is new. However, the idea of our proof is inspired by the paper by Barles and Georgelin [2] on fronts driven by Mean Curvature where they prove convergence for a scheme in the framework of discontinuous viscosity solutions (we also refer to Barles, Souganidis [4] for convergence in the framework of continuous viscosity solutions). Basically, it is sufficient to consider a test function touching the upper semicontinuous envelope of the numerical solution (obtained as $\Delta x, \Delta t$ go to zero) which violates the subsolution property and to derive from this some properties of the discrete solution for non-zero $\Delta x, \Delta t$. This corresponds to consider test functions touching the discrete analogue of the discontinuous function θ in order to get a contradiction with the basic properties of the algorithm.

The paper is organized as follows. In Section 2 we introduce our notation, present our GFMM algorithm and the main result of this paper, *i.e.* the convergence of the algorithm (Theorem 2.5). Several comments and the explanation of the subtleties of the algorithm are discussed in Section 3. Section 4 is devoted to prove comparison principles and symmetry for GFMM. In Section 5, several preliminary results are presented, focusing on properties of discrete times and on the geometry of the level sets of test functions. In Section 6, we use the results of Section 5 to prove the subsolution property of the *limsup** envelope of the numerical solution, while the comparison principle of Section 4 is used to prove this subsolution property at the initial time. The main result of Section 6 is the proof of our main Theorem (Theorem 2.5). Finally, in Section 7 we present some numerical simulations and comment these results in connection with our theoretical results.

2 The GFMM algorithm and the main result

In this section we give details for our GFMM algorithm for unsigned velocity. Let us start introducing our definitions and notation.

Let us consider a lattice $Q \equiv \{x_I = (x_{i_1}, .., x_{i_N}) = (i_1\Delta x, .., i_N\Delta x),\ I = (i_1, .., i_N) \in \mathbb{Z}^N\}$ with space step $\Delta x > 0$. We will also use a time step $\Delta t > 0$.

The following definitions will be useful in the following.

Definition 2.1 *The* **neighborhood of the node** $I \in \mathbb{Z}^N$ *is the set* $V(I) \equiv \{J \in \mathbb{Z}^N : |J - I| \leq 1\}$.

Definition 2.2 *Given the speed* $c_I^n \equiv c(x_I, t_n)$ *we define the function*

$$\widehat{c}_I^n \equiv \begin{cases} 0 & \textit{if there exists } J \in V(I) \textit{ such that } (c_I^n c_J^n < 0 \textit{ and } |c_I^n| \leq |c_J^n|), \\ c_I^n & \textit{otherwise.} \end{cases}$$

Definition 2.3 *The* **numerical boundary** ∂E *of a set* $E \subset \mathbb{Z}^N$ *is*

$$\partial E \equiv V(E) \backslash E$$

with

$$V(E) = \Big\{J \in \mathbb{Z}^N, \quad \exists I \in E, \quad J \in V(I)\Big\}$$

Definition 2.4 *Given a field* θ_I^n *with values* $+1$ *and* -1, *we define the two phases*

$$\Theta_\pm^n \equiv \{I : \theta_I^n = \pm 1\},$$

and the fronts

$$F_\pm^n \equiv \partial\Theta_\mp^n, \quad F^n \equiv F_+^n \cup F_-^n.$$

In the description of the algorithm we will use the following notations:

$$\pm g \geq 0 \text{ for } I \in F_{\pm} \tag{2.2}$$

means

$$+g \geq 0 \text{ for } I \in F_{+} \text{ and } -g \geq 0 \text{ for } I \in F_{-}. \tag{2.3}$$

Moreover,

$$\min_{\pm}\{0, g_{\pm}\} \equiv \min\{0, g_{+}, g_{-}\} \text{ and } \max_{\pm}\{0, g_{\pm}\} \equiv \max\{0, g_{+}, g_{-}\}. \tag{2.4}$$

2.1 The algorithm step-by-step

We describe now our GFMM algorithm for unsigned velocity. As one can see, to track correctly the evolution we need to introduce a discrete function $u_I^n \in \mathbb{R}^+$ defined for $I \in F^n$ to represents the approximated physical time for the front propagation at the nodes $I = (i_1, .., i_N)$ of the fronts at the n-th iteration of the algorithm.

Initialization

1. Set $n = 1$

2. *Initialize the field* θ^0 as
   $\theta_I^0 = \begin{cases} 1 & \text{for } x_I \in \Omega_0 \\ -1 & \text{elsewhere} \end{cases}$

3. *Initialize the time on* F^0
   $u_I^0 = 0$ for all $I \in F^0$

Main cycle

4. *Initialize* $\widehat{u}^{n-1}$ *everywhere on the grid*

$$\widehat{u}_{\pm,J}^{n-1} = \begin{cases} u_J^{n-1} & \text{for } J \in F_{\pm}^{n-1} \\ \infty & \text{elsewhere.} \end{cases}$$

5. *Compute* $\tilde{u}^{n-1}$ *on* F^{n-1} *as*

   Let $I \in F_{\pm}^{n-1}$, then

   (a) if $\pm\widehat{c}_I^{n-1} \geq 0$, $\tilde{u}_I^{n-1} = \infty$,

   (b) if $\pm\widehat{c}_I^{n-1} < 0$, we compute $\tilde{u}_I^{n-1}$ as the solution of the following second order equation:

$$\sum_{k=1}^{N} \left(\max_{\pm} \left(0, \tilde{u}_I^{n-1} - \widehat{u}_{+,I^{k,\pm}}^{n-1} \right) \right)^2 = \frac{(\Delta x)^2}{|\widehat{c}_I^{n-1}|^2} \quad \text{if } I \in F_{-}^{n-1},$$

(2.5)

$$\sum_{k=1}^{N}\left(\max_{\pm}\left(0,\tilde{u}_I^{n-1}-\widehat{u}_{-,I^{k,\pm}}^{n-1}\right)\right)^2=\frac{(\Delta x)^2}{|\widehat{c}_I^{n-1}|^2}\quad \text{if } I\in F_+^{n-1},$$

where

$$I^{k,\pm}=(i_1,..,i_{k-1},i_k\pm 1,i_{k+1},..,i_N).$$

6. $\tilde{t}_n=\min\left\{\tilde{u}_I^{n-1},\ I\in F^{n-1}\right\}$.
7. $\widehat{t}_n=\min\{\tilde{t}_n,t_{n-1}+\Delta t\}$.
8. $t_n=\max(t_{n-1},\widehat{t}_n)$
9. if $t_n=t_{n-1}+\Delta t$ and $t_n<\tilde{t}_n$ go to 4 with $n:=n+1$.
10. *Initialize the new accepted point*
    $NA_\pm^n=\{I\in F_\pm^{n-1},\ \tilde{u}_I^{n-1}=\tilde{t}_n\}$, $NA^n=NA_+^n\cup NA_+^n$
11. *Reinitialize* θ^n

$$\theta_I^n=\begin{cases}-1 & \text{for} \qquad I\in NA_+^n\\ 1 & \text{for} \qquad I\in NA_-^n\\ \theta_I^{n-1} & \text{elsewhere}\end{cases}$$

12. *Reinitialize* u^n *on* F^n

    (a) If $I\in F^n\backslash V(NA^n)$, then $u_I^n=u_I^{n-1}$.
    (b) If $I\in NA^n$ then $u_I^n=t_n$.
    (c) If $I\in(F^{n-1}\cap V(NA^n))\backslash NA^n$, then $u_I^n=u_I^{n-1}$.
    (d) If $I\in V(NA^n)\backslash F^{n-1}$ then $u_I^n=t_n$

13. Set $n:=n+1$ and go to 4

Let us describe a few features of this new algorithm:

1. We know, at each time step, the time on the fronts, *i.e.* on both side of the front. This is necessary to allow the changes of the velocity sign in time.
2. In *step 5*, we use the regularized velocity $\hat{c}$ and not c in order to stabilize the front.
3. *step 7* avoids large jumps in time and guarantees that $t_n-t_{n-1}\le\Delta t$ with Δt small enough.
4. *step 9* allows to increase the time. For example, if at time step n, we have $\widehat{c}_I^{n-1}=0\ \forall I\in F^{n-1}$, then there will not be new accepted points and the time will not change and the algorithm will be blocked without steps 7 and 9.
5. *step 8* guarantees that the physical time t_n does not decrease.

6. In *step 12*, for the reinitialization of u_I^n, we change its value only if a point of the neighborhood of the point I has been accepted. Moreover when u_I^n is updated, we use the physical time t_n and not $\tilde{t}_n$ or $\hat{t}_n$.

These choices, which can appear strange with respect to the classical FMM scheme, will be motivated in Section 3, giving also some examples which justify the new definition.

2.2 The main result

The scheme approximates the evolution of the fronts by a double *Narrow band* and the physical time by the sequence $\{t_k, k \in \mathbb{N}\}$, defined at the step 8 in the algorithm. Such sequence is nondecreasing and we can extract a subsequence $\{t_{k_n}, n \in \mathbb{N}\}$ strictly increasing such that

$$t_{k_n} = t_{k_n+1} = ... = t_{k_{n+1}-1} < t_{k_{n+1}}.$$

We denote by S_I^n the square cell $S_I^n = [x_I, x_I + \Delta x[\times[t_{k_n}, t_{k_{n+1}}[$ with

$$[x_I, x_I + \Delta x[= \Pi_{\alpha=1}^N [x_{i_\alpha}, x_{i_\alpha} + \Delta x[$$

and by ε the couple

$$\varepsilon = (\Delta x, \Delta t).$$

Let us define the following functions:

$$\theta^\varepsilon(x,t) = \begin{cases} \sup\{\theta_I^m : k_n \le m \le k_{n+1} - 1\} & \text{if } (x,t) \in S_I^n \text{ and } c(x_I, t_{k_n}) > 0 \\ \inf\{\theta_I^m : k_n \le m \le k_{n+1} - 1\} & \text{if } (x,t) \in S_I^n \text{ and } c(x_I, t_{k_n}) < 0 \\ \theta_I^m,\ \forall m :\ k_n \le m \le k_{n+1} - 1 & \text{if } (x,t) \in S_I^n \text{ and } c(x_I, t_{k_n}) = 0. \end{cases} \tag{2.6}$$

This definition is equivalent to the following

$$\theta^\varepsilon(x,t) = \theta_I^{k_{n+1}-1} \text{ if } (x,t) \in S_I^n. \tag{2.7}$$

We define the half-relaxed limits

$$\overline{\theta}^0(x,t) = \limsup_{\varepsilon\to 0, y\to x, s\to t} \theta^\varepsilon(y,s), \quad \underline{\theta}^0(x,t) = \liminf_{\varepsilon\to 0, y\to x, s\to t} \theta^\varepsilon(y,s). \tag{2.8}$$

We make the following assumption

(A) The velocity $c \in W^{1,\infty}(\mathbb{R}^N \times [0,T])$, for some constant $L > 0$ we have $|c(x',t') - c(x,t)| \le L(|x'-x| + |t'-t|)$, and Ω_0 is a C^2 open set, with bounded boundary $\partial\Omega_0$.

Theorem 2.5 (Convergence Result)
Under assumption (A), $\overline{\theta}^0$ (resp. $\underline{\theta}^0$) is a viscosity sub-solution (resp. super-solution) of (1.1). In particular, if (1.1) satisfies a comparison principle, then $\overline{\theta}^0 = (\underline{\theta}^0)^*$ and $(\overline{\theta}^0)_* = \underline{\theta}^0$ is the unique viscosity solution of (1.1).

Remark 2.6 *When the uniqueness holds, this is up to the upper and lower semi-continuous envelopes.*

Remark 2.7 *Note that when $c > 0$, our GFMM algorithm is a modified FMM algorithm where the time on the narrow band is computed using only the accepted points. In this monotone case the viscosity solution of (1.1) is unique and our result provides a convergence result (see also Test 3 in the last section).*

Remark 2.8 *The Lipschitz-continuity in time of the velocity could be relaxed to continuity, but is assumed here to simplify the presentation of the proofs, which are already quite complicated.*

3 Justifications and examples

In this section we will show that in the variable sign scheme it is necessary to introduce new variables to track correctly the evolution of the front.

3.1 Introduction of the numerical speed $\widehat{c}_I^n$

Let us show by an example in dimension $N = 1$ what would happen choosing c_I^n instead than $\widehat{c}_I^n$.
Consider the speed

$$c(x) = \begin{cases} -\delta & \text{if } x < x_I \\ \delta & \text{if } x \geq x_I \end{cases},$$

as plotted in Fig.2.

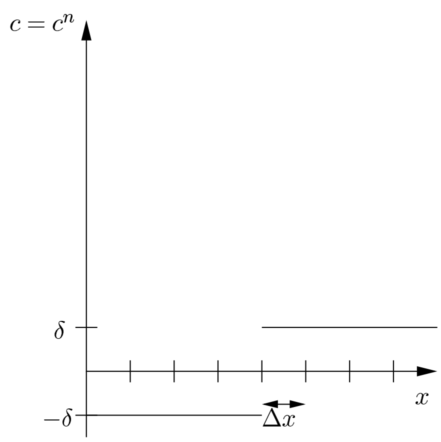

Figure 2: The velocity c^n.

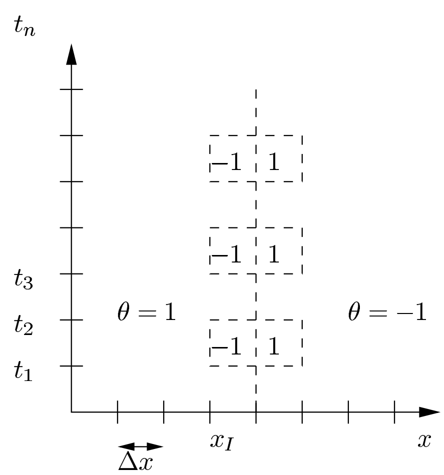

Figure 3: Evolution with the velocity c^n.

Suppose $\frac{\Delta x}{\delta} = \Delta t$, and $\theta_J^0 = 1$ for $J \leq I$, $\theta_J^0 = -1$ for $J > I$. Then the nodes I, $I+1$ will be accepted at the iteration 1, with $t_1 = \Delta t$. Then after each time interval Δt the nodes I and $I+1$ will change phase, producing spurious oscillations on the fronts, see Fig. 3.

Let us now consider the same example but with the numerical speed $\widehat{c}_I^n$, as plotted in Fig.4. In this case the nodes on the front have always speed zero, and the front does not move as one would expect.

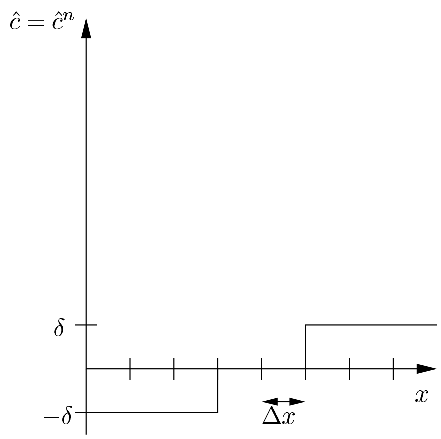

Figure 4: The velocity $\widehat{c}^n$.

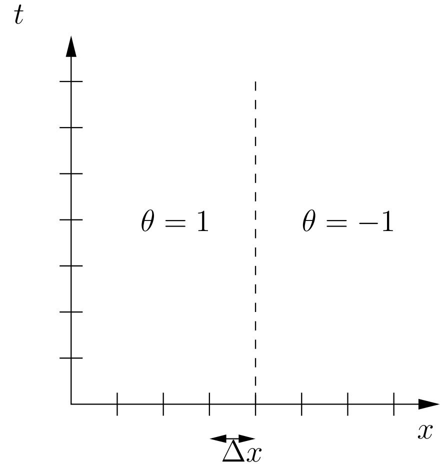

Figure 5: Evolution with the velocity $\widehat{c}^n$.

3.2 Introduction of the time step

In the case of monotone evolution with speed depending only on the space variables, the FMM algorithm approximates the corresponding stationary equation of (1.1) and then no time step Δt is required. In our more general case, the dependence of the speed in time makes necessary to introduce a time step Δt in the algorithm. We show its need by an example. We provide for simplicity the example in dimension $N = 1$.

Let us take as speed a linear function in time : $c(t) = T - t$, where T is a fixed positive constant. In this case the sequence of the discrete time $\{t_n\}_{n\in\mathbb{N}}$ is given by:

$$t_{n+1} = t_n + \frac{\Delta x}{|c(t_n)|}.$$

Let us define the sequence $\tau_n = T - t_n$. Such a sequence verifies: $\tau_n = f(\tau_{n-1})$ where $f(\tau) = \tau - \frac{\Delta x}{\tau}$. Since f is invertible for any $\tau > 0$, we then define $\tau_n = \Delta x$ and we evaluate τ_k by $\tau_k = f^{-1}(\tau_{k+1})$ for any $k < n$, then $\tau_k < \tau_{k-1}$.

Then we have defined

$$\begin{cases} t_m = T - \tau_m, & m = n, n-1, ..., n_0 \\ n_0 = \min\{m, & t_m \geq 0\}. \end{cases}$$

It results $t_n = T - \Delta x$ and then $t_{n+1} = T + 1 - \Delta x$, i.e. $t_{n+1} - t_n = 1$. We want to avoid this situations, since a big increment in the sequence of the discrete time step can bring a loss of informations and in general the algorithm would not converge to the correct evolution of the fronts. We show such a distribution of discrete time together with the linear speed in Fig.6 and we show the wrong evolution of a front in this case in Fig.7.

If instead we introduce a threshold Δt as in the GFMM algorithm step 7, then it results $t_{n+1} = t_n + \Delta t$, and we get the correct evolution for Δt small enough (see Fig. 8).

3.3 Why we update the front using t instead of $\widehat{t}$

We explain by an example in dimension $N = 1$, why it is correct to assign the value t_n instead of $\widehat{t}_n$ on the front F^n in the Step 12 of the algorithm.

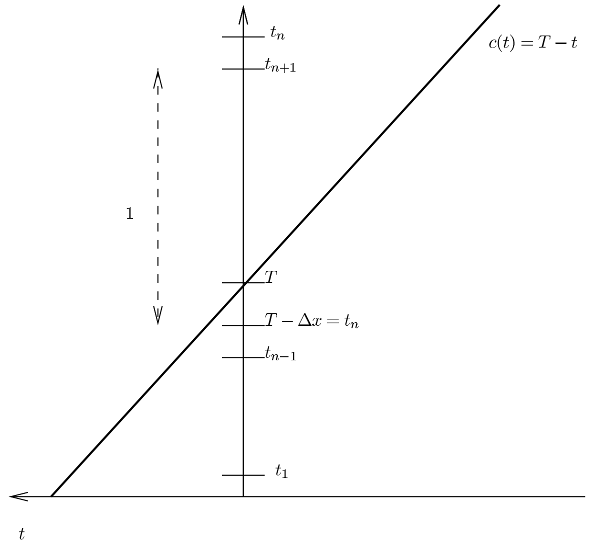

Figure 6: Jump in the discrete time without threshold Δt in the case of a linear in time velocity

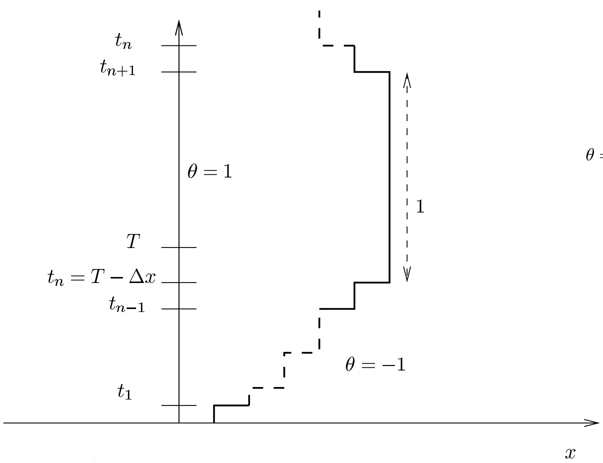

Figure 7: Front evolution without the threshold Δt

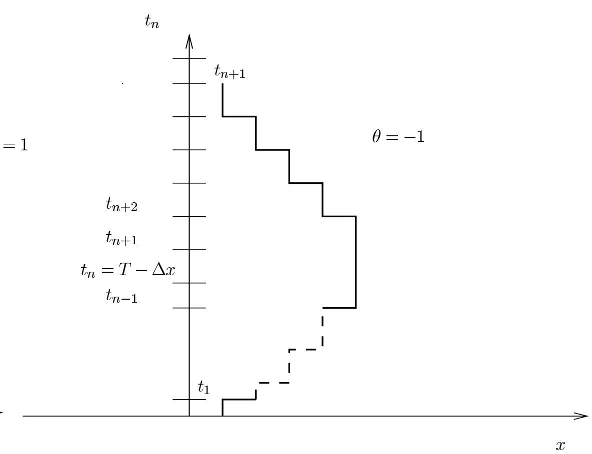

Figure 8: Front evolution with the threshold Δt

Fixed $p \in \mathbb{N}$, we define $\Delta s = p\Delta t$ and $\delta = \frac{\Delta x}{\Delta t}$ and we consider a piecewise constant speed for $t \geq 0$:

$$c(t) = \begin{cases} \delta & \text{if} \quad t \in [(2k-1)\Delta s, 2k\Delta s) \text{ for some } k \in \mathbb{N}\backslash\{0\}, \\ 0 & \text{otherwise} \end{cases}$$

as in Fig.9 on the left.

If we update the front using $\widehat{t}$ instead of t, the corresponding evolution will be the one plotted on the right of Fig.9. In fact the front will start to move using the velocity given at time $t_p = p\Delta t$ and since $\tilde{t}_{p+1} = 0 + \frac{\Delta x}{c(t_p)} = \frac{\Delta x}{c(t_p)} = \Delta t$, then $\widehat{t}_{p+1} = \min\{\tilde{t}_{p+1}, t_p + \Delta t\} = \min\{\Delta t, \Delta s + \Delta t\} = \Delta t$ and $t_{p+1} = \max(t_p, \widehat{t}_{p+1}) = \max(\Delta s, \Delta t) = \Delta s = t_p$. So the front will propagate on the right at the iteration $p, ..., 2p-1$ but at constant time $t = t_p$ (Fig.9). On the contrary, if we update the front using t, the front will start to move using the velocity given at time $t_p = p\Delta t$ and the iteration $p+1$ will be as before. But since $\tilde{t}_{p+2} =$

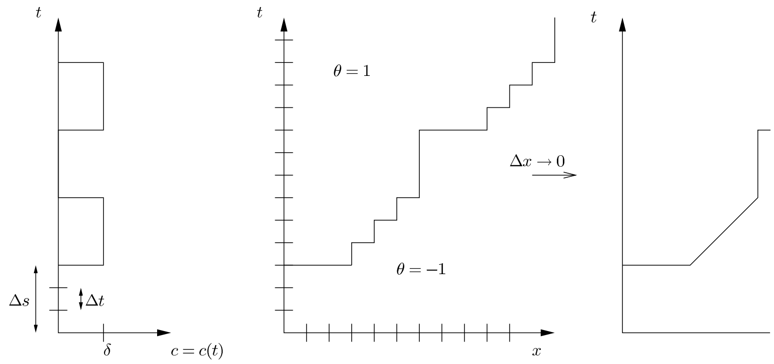

Figure 9: Wrong evolution when the time is updated with $\hat{t}$.

$t_{p+1}+\frac{\Delta x}{c(t_p)} = \Delta s+\frac{\Delta x}{c(t_p)} = \Delta s+\Delta t$, then $\widehat{t}_{p+2} = \min\{\tilde{t}_{p+2}, t_{p+1}+\Delta t\} = \min\{\Delta t+\Delta s, \Delta t+\Delta s\}$ and $t_{p+2} = \max(t_{p+1}, \widehat{t}_{p+1}) = \max(\Delta s + \Delta t, \Delta t) = \Delta s + \Delta t$. So the front will propagate on the right at the iteration $p, ..., 2p-1$ but at linear times $t_m = m\Delta t$ (Fig.10).

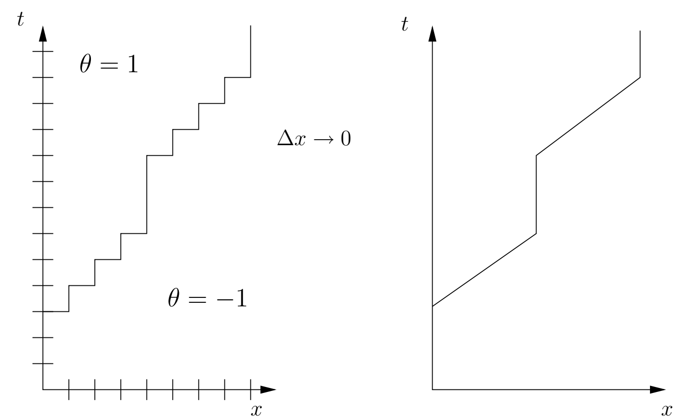

Figure 10: Correct evolution when the time is updated with t.

4 Comparison principles for the GFMM algorithm

As we said in the introduction, our convergence result will be proved in the framework of discontinuous viscosity solutions. To this end the role of comparison principles is crucial.

In this Section, we first present a property of symmetry of the algorithm, then present some comparison principles in some special cases and finally a counter-example to a general comparison principle.

This Section shows in particular that statements on our GFMM algorithm are highly non-trivial in general.

4.1 Symmetry of the algorithm

The following lemma claims that if we change the sign of the velocity and the sign of the two phases at the initial time, then the GFMM algorithm computes the same front.

Lemma 4.1 (Symmetry of the GFMM algorithm)
We denote by $\overline{\theta}^0[\theta^0, c]$ *and* $\underline{\theta}^0[\theta^0, c]$ *the functions constructed by the GFMM algorithm with initial condition* θ^0 *and velocity* c*. Then we have*

$$\underline{\theta}^0[\theta^0, c] = -\overline{\theta}^0[-\theta^0, -c].$$

Proof of Lemma 4.1
With the same kind of notation, we remark that

$$\theta_I^n[-\theta^0, -c] = -\theta_I^n[\theta^0, c].$$

We then have, for $x \in [x_I, x_I + \Delta x[,\ t \in [t_{k_n}, t_{k_{n+1}}[$ and $c(x_I, t_{k_n}) > 0$

$$\begin{aligned}\theta^\varepsilon[\theta^0, c](x,t) &= \sup\{\theta_I^k[\theta^0, c],\ k_n \le k \le k_{n+1} - 1\}\\ &= -\inf\{-\theta_I^k[\theta^0, c],\ k_n \le k \le k_{n+1} - 1\}\\ &= -\inf\{\theta_I^k[-\theta^0, -c],\ k_n \le k \le k_{n+1} - 1\}\\ &= -\theta^\varepsilon[-\theta^0, -c].\end{aligned}$$

The result is similar for $c(x_I, t_{k_n}) \le 0$. Therefore

$$\underline{\theta}^0[\theta^0, c] = -\overline{\theta}^0[-\theta^0, -c].$$

□

4.2 Comparison principles

Proposition 4.2 (Comparison principle for the time)
We denote by u_I^n *(resp.* v_I^n*) the numerical solution at the point* (x_I, t_n) *of the GFMM algorithm with velocity* c_u *(resp.* c_v*). We assume that there exists* $T > 0$ *such that for all* $(x,t) \in \mathbb{R}^N \times [0, T]$

$$\inf_{s\in[t-\Delta t, t],\ s\ge 0} c_v(x,s) \ge \sup_{s\in[t-\Delta t, t],\ s\ge 0} (c_u(x,s))^+$$

where $(f)^+$ *is the positive part of* f*. We assume that*

$$\{\theta_u^0 = 1\} \subset \{\theta_v^0 = 1\} \quad \text{and} \quad v^0 = u^0 = 0.$$

We define $\bar{m}$ *and* $\bar{k}$ *such that*

$$\begin{cases} t_{\bar{m}} \le T < t_{\bar{m}+1} \\ s_{\bar{k}} \le T < s_{\bar{k}+1} \end{cases}$$

where $(t_m)_m$ *and* $(s_m)_m$ *are respectively the sequence of time constructed by the GFMM algorithm with velocity* c_u *and* c_v*. We then consider*

$$v_I = \begin{cases} v_I^0 & \text{if} \quad \theta_{v,I}^0 = 1 \\ v_I^k & \text{if} \quad I \in NA_v^k \quad \text{for some} \quad k \le \bar{k} + 1 \\ s_{\bar{k}+1} & \text{if} \quad \theta_{v,I}^{\bar{k}} = -1 \end{cases}$$

Then, $\forall l \le \bar{m}$*,* $\forall I \in NA_u^l$*, we have*

$$v_I \le u_I^l.$$

Remark 4.3 *Here the notation for $\theta_u, \theta_v, NA_u^l$ and further notation in the sequel are obvious and are not explained. Moreover we also remark that the front for v passes at most one time at a given point because $c_v \geq 0$.*

Proof of Proposition 4.2
We argue by contradiction. We denote by $m(u)$ the first index such that there exists $I \in NA_u^{m(u)}$ such that

$$u_I^{m(u)} < v_I \tag{4.9}$$

We define

$$k(v) \quad \text{such that} \quad I \in NA_v^{k(v)}$$

with the convention that $k(v) = \bar{k} + 1$ if $\theta_{v,I}^{\bar{k}} = -1$. This implies that

$$t_{m(u)} = u_I^{m(u)} < v_I = s_{k(v)}$$

The proof distinguishes two cases.

1. $\boldsymbol{I \in NA_{-,u}^{m(u)} \subset F_{-,u}^{m(u)-1}}$**.**
   We claim that for all $J \in V(I) \backslash \{I\}$, we have

$$\widehat{u}_{+,J}^{m(u)-1} \geq \widehat{v}_{+,J}^{k(v)-1} \tag{4.10}$$

   Indeed assume that $\widehat{u}_{+,J}^{m(u)-1} < \infty$ (if $\widehat{u}_{+,J}^{m(u)-1} = \infty$, then (4.10) holds), then $J \in F_{+,u}^{m(u)-1}$ and we have

$$t_{m(u)} \geq \widehat{u}_{+,J}^{m(u)-1} \geq v_J.$$

   It just remains to show that $v_J = \widehat{v}_{+,J}^{k(v)-1}$. We argue by contradiction. Assume that $\widehat{v}_{+,J}^{k(v)-1} = \infty$, *i.e* $J \in \{\theta_v^{k(v)-1} = -1\}$. Then $v_J \geq s_{k(v)}$. This contradicts the fact that $v_J \leq t_{m(u)} < s_{k(v)}$ and proves (4.10).

   We define

$$k^* := \sup\{k, s_k \leq t_{m(u)}\} < k(v).$$

   In particular, we have $t_{m(u)} - \Delta t \leq s_{k^*} \leq t_{m(u)}$. Since for all $J \in V(I) \cap F_{+,u}^{m(u)-1}$

$$t_{m(u)} \geq \widehat{u}_{+,J}^{m(u)-1} \geq \widehat{v}_{+,J}^{k(v)-1}$$

   we deduce that

$$s_{k^*} \geq \widehat{v}_{+,J}^{k(v)-1}. \tag{4.11}$$

   Indeed, $+\infty > \widehat{v}_{+,J}^{k(v)-1} > s_{k^*}$ would imply that there exists $k' > k^*$ such that $t_{m(u)} \geq \widehat{v}_{+,J}^{k(v)-1} = s_{k'}$ which contradicts the definition of k^*.
   Then we claim that for all $J \in V(I) \cap F_{+,u}^{m(u)-1}$

$$\widehat{v}_{+,J}^{k(v)-1} = \widehat{v}_{+,J}^{k^*}. \tag{4.12}$$

We now prove the claim (4.12). First, because we have $\widehat{v}_{+,J}^{k(v)-1} < +\infty$, we deduce that $\theta_{v,J}^{k(v)-1} = 1$ and then there exists $k \le k(v)-1$ such that if $k \ge 1$, then $J \in NA_v^k$ and $\widehat{v}_{+,J}^{k(v)-1} = v_J^k = s_k$, and if $k = 0$, then $\theta_{v,J}^0 = 1$ and $\widehat{v}_{+,J}^{k(v)-1} = v_J^0 = 0$.
Assume by contradiction that $k > k^*$. Then

$$\widehat{v}_{+,J}^{k(v)-1} = v_J^k = s_k \ge s_{k^*+1} > s_{k^*}$$

Contradiction with (4.11). Therefore $k \le k^*$. Now we have $\theta_{v,I}^{k(v)} = 1$ and $\theta_{v,I}^m = -1$ for $m \le k(v)-1$. Therefore $J \in F_{+,v}^{k^*}$ and

$$\widehat{v}_{+,J}^{k(v)-1} = v_J^k = \widehat{v}_{+,J}^{k^*}$$

which ends the proof of the claim (4.12). We deduce that

$$\widehat{v}_{+,J}^{k(v)-1} = \widehat{v}_{+,J}^{k^*} \le \widehat{u}_{+,J}^{m(u)-1},$$

where we have used (4.10). We define the following function

$$f_{\widehat{u}^m}^2(t) = \sum_{k=1}^{N} \left(\max_{\pm} \left(0, t - \widehat{u}_{+,I^{k,\pm}}^m\right)\right)^2.$$

We then have, using the fact that $\tilde{v}_I^{k^*} \ge s_{k^*+1} > s_{k^*}$

$$f_{\widehat{v}^{k^*}}(s_{k^*+1}) \le f_{\widehat{v}^{k^*}}(\tilde{v}_I^{k^*}) = \left| \frac{\Delta x}{\widehat{c}_{I,v}^{k^*}} \right| \le \left| \frac{\Delta x}{\widehat{c}_{I,u}^{m(u)-1}} \right| = f_{\widehat{u}^{m(u)-1}}(\tilde{u}_I^{m(u)-1}) \le f_{\widehat{v}^{k^*}}(\tilde{u}_I^{m(u)-1})$$

We then deduce that

$$s_{k^*+1} \le \tilde{u}_I^{m(u)-1} \le u_I^{m(u)} = t_{m(u)}.$$

This is absurd.
□

2. $\boldsymbol{I \in NA_{+,u}^{m(u)} \subset F_{+,u}^{m(u)-1}}$.
   We consider the following subcases
   (a) $I \in \{\theta_v^0 = 1\}$. Then $v_I = v_I^0 = 0 = u_I^0 \le u_I^{m(u)}$. This is absurd.
   (b) $I \in \{\theta_v^0 = -1\}$. Then $\theta_{u,I}^0 = -1$ and so there exists $n < m(u)$ such that

$$\theta_{u,I}^{n-1} = -1 \quad \text{and} \quad \theta_{u,I}^n = 1.$$

   This implies that

$$u_I^n \ge v_I > u_I^{m(u)} \ge u_I^n.$$

   This is absurd. □

Remark 4.4 *If we implicit the computation of the gradient, i.e. the computation of $\tilde{u}$ in step 5, the situation seems better and one could expect to prove a general comparison principle without restriction on the velocity.*

We now rephrase this comparison principle for the functions θ^ε and prove it.

Corollary 4.5 (Comparison principle with a nonnegative velocity)
Under the assumptions of Proposition 4.2, we have for all $(x,t) \in \mathbb{R}^N \times [0,T]$

$$\theta_u^\varepsilon(x,t) \leq \theta_v^\varepsilon(x,t).$$

Proof of Corollary 4.5
By contradiction, assume that there exist x_I and t such that

$$\theta_u^\varepsilon(x_I,t) = 1 \quad \text{and} \quad \theta_v^\varepsilon(x_I,t) = -1. \tag{4.13}$$

We denote by t the first time such that (4.13) holds. We then have, since $c_v \geq 0$,

$$\theta_u^\varepsilon(x_I,s) = -1 \quad \text{if} \quad s < t.$$

We then deduce that there exists $m(u)$ such that $t = t_{m(u)}$, $I \in NA_u^{m(u)}$ and $u_I^{m(u)} = t_{m(u)} = t$. Moreover, since the index I has not been already accepted for v, we have $v_I > t = u_I^{m(u)}$. This is absurd. $\square$

Corollary 4.6 (Comparison principle for a nonpositive velocity)
We denote by u_I^n (resp. v_I^n) the numerical solution at the point (x_I,t_n) of the GFMM algorithm with velocity c_u (resp. c_v). We assume that there exists $T > 0$ such that for all $(x,t) \in \mathbb{R}^N \times [0,T]$

$$\sup_{s\in[t-\Delta t,t],\ s\geq 0} c_u(x,s) \leq \inf_{s\in[t-\Delta t,t],\ s\geq 0} -(c_v(x,s))^-$$

where $(f)^- \geq 0$ is the negative part of f. We assume that

$$\{\theta_v^0 = -1\} \subset \{\theta_u^0 = -1\} \quad \text{and} \quad v^0 = u^0 = 0.$$

Then, for all $(x,t) \in \mathbb{R}^N \times [0,T]$, we have

$$\theta_u^\varepsilon(x,t) \leq \theta_v^\varepsilon(x,t).$$

Proof of Corollary 4.6
This is a straightforward consequence of Corollary 4.5 and the fact that $\theta^\varepsilon[-\theta^0,-c] = -\theta^\varepsilon[\theta^0,c]$ (with the notation of Lemma 4.1). $\square$

4.2.1 Counter-example for the comparison principle in general

We now give a counter-example for a more general comparison principle for which the two velocities can change their signs.

Proposition 4.7 (Counter-example)
Let $N = 1$. We assume that the velocity c_u and c_v are null everywhere except on a node I for which $c_u(x_I,\cdot) \geq c_v(x_I,\cdot)$ are given in Figure 11 and 12 respectively with $\dfrac{\Delta x}{\delta} = k\Delta t$. We also suppose that

$$\theta_{u,J}^0 = 1 \text{ if and only if } J \leq I \quad \text{and} \quad \theta_{v,J}^0 = 1 \text{ if and only if } J < I.$$

Then

$$\theta_{u,I}^{k+3} = -1 \quad \text{and} \quad \theta_{v,I}^{k+3} = 1.$$

Proof of Proposition 4.7

1. For the GFMM associated to u.
   The node I will be accepted with a time $u_I^k = t_k = k\Delta t$ and we will affect the value $t_k = k\Delta t$ to u_I^k. Then the velocity will change of sign and the node I will be accepted again with a time $2k\Delta t$ (see Figure 11).

2. For the GFMM associated to v.
   Since the velocity is nonpositive, nothing will move. The current time t_n will continue to increase and we will have $t_{k+2} = (k+2)\Delta t$ and then the velocity will become positive. The node I will then be accepted with a time $v_I^{k+3} = t_{k+2} = (k+2)\Delta t$ (see Figure 12).

We then conclude that the node I will be accepted for v before to be accepted for u and so no comparison principle can hold in general.

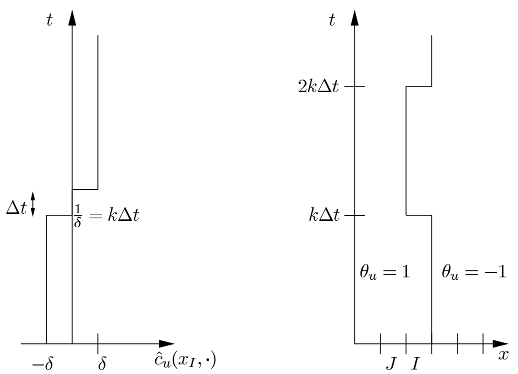

Figure 11: Velocity and evolution of the front for u.

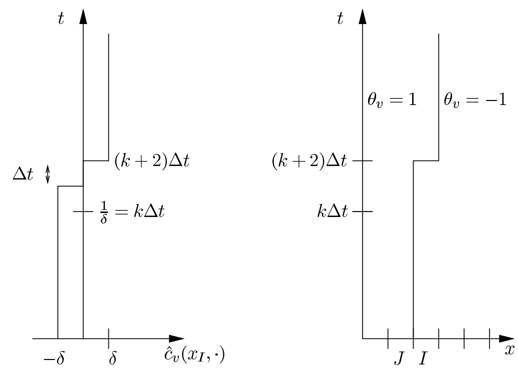

Figure 12: Velocity and evolution of the front for v.

5 Preliminary results on the discrete time and on the level sets of test functions

The GFMM algorithm described in Section 2 has several properties which fit the physics of the problem we want to solve. We present in this Section several results that will be used in the proof of Proposition 6.1 which is crucial for the proof of our main result of convergence.

In a first subsection, we present some properties of the various times $\hat{u}, t, \tilde{t}$ appearing in our algorithm, and in a second subsection we give some geometrical consequences of the existence of test functions tangent from above to our function θ^ε.

5.1 Preliminary results on the discrete time

Lemma 5.1 (Time character of the $\widehat{u}$)
Assume there exists $\delta > 0$ and $(I, n) \in \mathbb{Z}^N \times \mathbb{N}$ such that $c(x_I, t_n) \geq \delta > 0$, $\theta_I^{n-1} = -1$ and $\theta_I^n = 1$ (resp. $c(x_I, t_n) \leq -\delta < 0$, $\theta_I^{n-1} = 1$ and $\theta_I^n = -1$), then for any $J \in V(I) \cap F_+^{n-1}$

(resp. $J \in V(I) \cap F_-^{n-1}$), we have for $\Delta x \leq \dfrac{\delta^2}{16L}$

$$\widehat{u}_{+,J}^{n-1} = \sup\{t_m \leq t_{n-1},\ \theta_J^{m-1} = -1, \theta_J^p = 1, \quad \textit{for} \quad m \leq p \leq n-1\} \quad > \quad t_n - \frac{4\Delta x}{\delta}$$

with the convention that $\widehat{u}_{+,J}^{n-1} = 0$ *if* $\theta_J^p = 1$ *for* $0 \leq p \leq n-1$

$$(\textit{resp. } \widehat{u}_{-,J}^{n-1} = \sup\{t_m \leq t_{n-1},\ \theta_J^{m-1} = 1, \theta_J^p = -1, \quad \textit{for} \quad m \leq p \leq n-1\} \quad > \quad t_n - \frac{4\Delta x}{\delta}.$$

with the convention that $\widehat{u}_{+,J}^{n-1} = 0$ *if* $\theta_J^p = -1$ *for* $0 \leq p \leq n-1$*).*

This lemma claims in fact that the $\widehat{u}_{+,J}^{n-1}$ is defined as the last time at which the front passed through J. Intuitively, this comes from the fact that, since the velocity is locally non-negative and since the front has crossed the node x_I at time t_n, it has crossed the node x_J at a time closed to t_n.

Proof of Lemma 5.1

We only do the proof in the case $c > 0$ (the case $c < 0$ is similar). By assumptions, c is Lipschitz-continuous with constant L, and there exists $\delta_0 \leq \delta/(4L)$ such that for all $(x_J, t_m) \in B_{\delta_0}(x_I) \times [t_n - \delta_0, t_n + \delta_0]$, we have

$$\widehat{c}_J^m \geq \frac{\delta}{2}.$$

This implies that

$$\theta_I^m = -1 \text{ for all } m \text{ such that } \ t_n - \delta_0 \leq t_m \leq t_{n-1}. \tag{5.14}$$

Let $J \in V(I) \cap F_+^{n-1}$. We define

$$m_J = \sup\{m \leq n-1,\ \theta_J^{m-1} = -1, \theta_J^m = 1\}.$$

We claim that for all $J \in V(I) \cap F_+^{n-1}$, we have $t_{m_J} > t_n - \delta_0$ for Δx small enough. Indeed, by contradiction, assume that there exists $J \in V(I) \cap F_+^{n-1}$ such that $t_{m_J} \leq t_n - \delta_0$. Let us define $p \geq 0$ such that

$$t_n = ... = t_{n-p} > t_{n-p-1}.$$

We then have $\widehat{u}_{+,J}^{n-p-1} \leq t_n - \delta_0$ and $\tilde{u}_I^{n-p-1} \geq t_{n-p} = t_n$. Using the fact that

$$\sum_{k=1}^{N} \left(\max_{\pm} \left(0, \tilde{u}_I^{n-p-1} - \widehat{u}_{+,I^{k,\pm}}^{n-p-1}\right)\right)^2 = \left(\frac{\Delta x}{\widehat{c}_I^{n-p-1}}\right)^2$$

we then deduce that

$$\delta_0 = t_{n-p} - (t_n - \delta_0) \leq \tilde{u}_I^{n-p-1} - \widehat{u}_J^{n-p-1} \leq \frac{2\Delta x}{\delta}.$$

This is absurd for the choice $\delta_0 = \dfrac{4\Delta x}{\delta} \leq \dfrac{\delta}{4L}$ which is valid for Δx small enough. Moreover, using (5.14), we deduce that $J \in F^m$ for all $m_J \leq m \leq n-1$. This implies that $\widehat{u}_{+,J}^{n-1} = u_J^{n-1} = u_J^{m_J} = t_{m_J}$. $\square$

The following lemma is concerned with the fact that we can control the decay of the time $\tilde{t}_n$ given by the GFMM algorithm, by the variations in time of the velocity.

Lemma 5.2 (Error estimate between t_n and $\tilde{t}_n$)
Assume that there exists $I \in NA^n$ such that $|\widehat{c}_I^{n-1}| \geq \delta > 0$. Then, the following estimate holds

$$\left(t_n - \tilde{t}_n\right)^+ \leq \frac{2L}{\delta^2}\Delta x \Delta t \quad \text{if} \quad \Delta t \leq \frac{\delta}{2L}$$

Proof of Lemma 5.2
We only treat the case $c_I^{n-1} \geq \delta > 0$ (the other case is similar). Assume that $\tilde{t}_n < t_n$, then necessarily $t_n = t_{n-1}$. We define $p > 0$ such that

$$t_{n-p-1} < t_{n-p} = ... = t_{n-1} = t_n.$$

In particular, we have

$$t_{n-p} \leq \tilde{t}_{n-p} \leq \tilde{u}_J^{n-p-1} \quad \forall J \in F^{n-p-1}$$

and

$$\tilde{t}_n = \tilde{u}_I^{n-1} \leq \tilde{u}_J^{n-1} \quad \forall J \in F^{n-1}.$$

We claim that $I \in F_-^{n-p-1}$. Indeed, assume that $I \notin F_-^{n-p-1}$. Using the fact that $\theta_I^{n-p-1} = -1$ (since $\widehat{c}_I > 0$), we deduce that for all $J \in V(I) \cap F_+^{n-1}$, we have $\theta_J^{n-p-1} = -1$ and so $\widehat{u}_{+,J}^{n-1} = t_n$, this means that also the node J has been accepted at the physical time t_n . This implies that $\tilde{u}_I^{n-1} > t_n$ and this is absurd.

Moreover, because $t_{n-p} - t_{n-p-1} \leq \Delta t$, we have $\widehat{c}_I^{n-p-1} \geq \frac{\delta}{2}$ for $\Delta t \leq \dfrac{\delta}{2L}$. We then have

$$\sum_{k=1}^{N} \left(\max_{\pm} \left(0, \tilde{u}_I^{n-p-1} - \widehat{u}_{+,I^{k,\pm}}^{n-p-1}\right)\right)^2 = \left(\frac{\Delta x}{\widehat{c}_I^{n-p-1}}\right)^2 \tag{5.15}$$

and

$$\sum_{k=1}^{N} \left(\max_{\pm} \left(0, \tilde{u}_I^{n-1} - \widehat{u}_{+,I^{k,\pm}}^{n-1}\right)\right)^2 = \left(\frac{\Delta x}{\widehat{c}_I^{n-1}}\right)^2. \tag{5.16}$$

Let us compare $\widehat{u}_{+,J}^{n-1}$ and $\widehat{u}_{+,J}^{n-p-1}$ for $J \in V(I) \cap F_+^{n-1}$. If $J \notin F_+^{n-p-1}$, then u_J changes values during the iterations $n - p \leq m \leq n-1$, and for such m we have $\widehat{u}_{+,J}^{n-1} = u_J^m = t_m = t_n$. Since $\tilde{t}_n < t_n$, then this node $J \in V(I)$ does not contribute to the evaluation of (5.16) and

$$\sum_{k=1}^{N} \left(\max_{\pm} \left(0, \tilde{t}_n - \widehat{u}_{+,I^{k,\pm}}^{n-p-1}\right)\right)^2 = \sum_{k=1}^{N} \left(\max_{\pm} \left(0, \tilde{t}_n - \widehat{u}_{+,I^{k,\pm}}^{n-1}\right)\right)^2. \tag{5.17}$$

Let us denote by

$$f_q(v) = \left\{ \sum_{k=1}^{N} \left(\max_{\pm} \left(0, v - \widehat{u}_{+,I^{k,\pm}}^{q}\right)\right)^2 \right\}^{1/2}.$$

The function f_q verifies for any $q \in \mathbb{N}$ such that $I \in F_-^q$:

$$f_q(\tilde{u}_I^q) = \frac{\Delta x}{|\widehat{c}_I^q|}, \quad f_q'(v) \geq 1.$$

Then

$$
\begin{aligned}
t_n - \tilde{t}_n \leq \tilde{u}_I^{n-p-1} - \tilde{t}_n &\leq f_{n-p-1}(\tilde{u}_I^{n-p-1}) - f_{n-p-1}(\tilde{t}_n) \\
&= f_{n-p-1}(\tilde{u}_I^{n-p-1}) - f_{n-1}(\tilde{t}_n) = \Delta x \left(\frac{1}{|\widehat{c}_I^{n-p-1}|} - \frac{1}{|\widehat{c}_I^{n-1}|} \right) \\
&\leq \Delta x \frac{|\widehat{c}_I^{n-p-1} - \widehat{c}_I^{n-p}|}{|\widehat{c}_I^{n-p-1}||\widehat{c}_I^{n-p}|} \\
&\leq \frac{\Delta x |\partial_t c|_{L^\infty} |t_{n-p} - t_{n-p-1}|}{|\widehat{c}_I^{n-p-1}||\widehat{c}_I^{n-p}|} \\
&\leq \frac{2\Delta x |\partial_t c|_{L^\infty} \Delta t}{\delta^2}.
\end{aligned}
$$

□

5.2 Preliminary results on the level sets of test functions

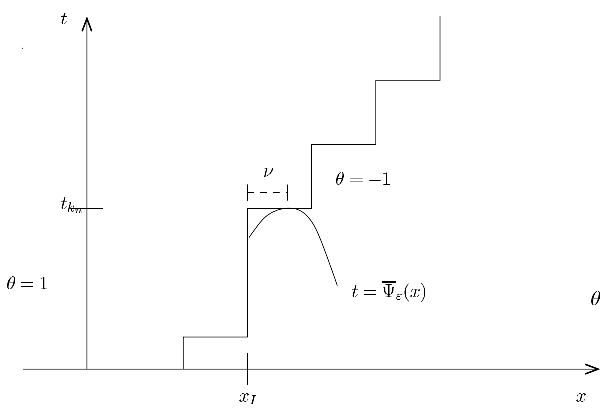

Figure 13: Test function from below

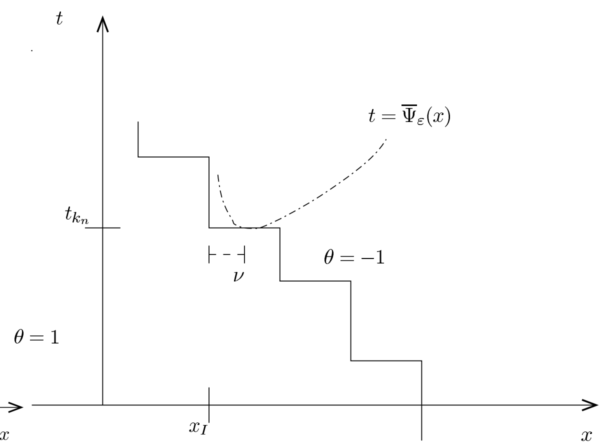

Figure 14: Test function from above

Lemma 5.3 (Separation of the phases of θ^ε by the level set of a test function) *Let $\varphi \in C^2$ in a neighborhood V of (x_0, t_0) such that $\varphi_t(x_0, t_0) > 0$ (resp. $\varphi_t(x_0, t_0) < 0$). There exist $\delta_0 > 0$, $r > 0$, $\tau > 0$ such that if $\max\limits_{\bar{V}}((\theta^\varepsilon)^* - \varphi)$ is reached at $(x_\varepsilon, t_\varepsilon) \in B_{\delta_0}(x_0, t_0) \subset V$ with $(\theta^\varepsilon)^*(x_\varepsilon, t_\varepsilon) = 1$, then there exists $\Psi_\varepsilon \in C^2(B_r(x_0), (t_0 - \tau, t_0 + \tau))$ such that*

(i) For all $(x_J, t_m) \in Q_{r,\tau}(x_0, t_0) = B_r(x_0) \times (t_0 - \tau, t_0 + \tau)$

$$\theta^\varepsilon(x_J, t_m) = 1 \quad \Longrightarrow \quad t_m \geq \Psi_\varepsilon(x_J) \text{ (resp. } t_m \leq \Psi_\varepsilon(x_J)).$$

(ii) There exists $(I, n) \in \mathbb{Z}^N \times \mathbb{N}$ such that

$$(x_\varepsilon, t_\varepsilon) \in \bar{Q}_I^n = [x_I, x_I + \Delta x] \times [t_{k_n}, t_{k_{n+1}}], \quad (\theta^\varepsilon)^*(x_I, t_{k_n}) = 1, \quad t_{k_n} = \Psi_\varepsilon(x_I)$$

and

$$\theta_I^{\overline{n}} = 1, \quad \theta_I^m = -1 \quad m_0 \le m \le \overline{n} - 1$$

$$\big(\textit{resp. } \theta_I^{\overline{n}} = -1, \quad \theta_I^m = 1 \quad m_0 \le m \le \overline{n} - 1\big)$$

where

$$\overline{n} = \inf\big\{k, \quad k_n \le k \le k_{n+1} - 1, \quad \theta_I^k = 1 \quad (\textit{resp.} \quad \theta_I^k = -1)\big\}$$

and $m_0 = \inf\{m,\ t_m \ge t_0 - \tau\}$.

(iii) The following Taylor expansion holds

$$\Psi_\varepsilon(x_J) = \Psi_\varepsilon(x_I) - \frac{D\varphi(x_0,t_0)}{\varphi_t(x_0,t_0)}(x_J - x_I) + (\Delta x)\ O\left(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|\right).$$

(iv) If $\varphi_t(x_0,t_0) < 0$*, then for all* $(x_J, t_{k_n}) \in Q_{r,\tau}(x_0,t_0) = B_r(x_0) \times (t_0 - \tau, t_0 + \tau)$

$$\theta^\varepsilon(x_J, t_{k_{n-1}}) = 1 \quad \textit{and} \quad \theta^\varepsilon(x_J, t_{k_n}) = -1 \quad \Longrightarrow \quad t_{k_n} \le \Psi_\varepsilon(x_J).$$

Proof of Lemma 5.3

We consider the case $\varphi_t(x_0,t_0) > 0$. The other case can be treated in a similar way. We define $\varphi_\varepsilon = \varphi + ((\theta^\varepsilon)^* - \varphi)(x_\varepsilon, t_\varepsilon)$. In particular, we have $(\theta^\varepsilon)^* \le \varphi_\varepsilon$ and $(\theta^\varepsilon)^*(x_\varepsilon, t_\varepsilon) = \varphi_\varepsilon(x_\varepsilon,t_\varepsilon) = 1$. We start by proving (i) and (ii). The proof is decomposed in several steps.

Step 1. We have $t_\varepsilon = t_{k_n}$.

Indeed, assume that $t_\varepsilon \in (t_{k_n}, t_{k_{n+1}})$. Using the fact that $(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon) = 1$, we deduce that $(\theta^\varepsilon)^*(x_\varepsilon, t) = 1$ for $t_{k_n} \le t \le t_{k_{n+1}}$ and so $\varphi_t(x_\varepsilon, t_\varepsilon) = 0$. This is absurd for δ_0 small enough since $\varphi_t(x_0,t_0) > 0$.

Step 2. We have $(\theta^\varepsilon)^* = -1$ on all $Q_J^{n-1} =]x_J, x_J + \Delta x[\times]t_{k_{n-1}}, t_{k_n}[$ such that $(x_\varepsilon, t_{k_n}) \in \bar{Q}_J^{n-1}$.

Indeed, since $\varphi_\varepsilon(x_\varepsilon, t_{k_n}) = 1$ and $(\varphi_\varepsilon)_t > 0$, we deduce that $\varphi_\varepsilon(x_\varepsilon, t) < 1$ if $t < t_{k_n}$. Using the fact that $(\theta^\varepsilon)^* - \varphi_\varepsilon$ reaches a maximum in (x_ε, t_{k_n}), yields

$$(\theta^\varepsilon)^*(x_\varepsilon, t) \le \varphi_\varepsilon(x_\varepsilon, t) < 1 \quad \text{if } t < t_{k_n}$$

and so

$$(\theta^\varepsilon)^*(x_\varepsilon, t) = -1 \quad \text{if } t < t_{k_n}.$$

Using the semi-continuity of $(\theta^\varepsilon)^*$, one deduce that

$$(\theta^\varepsilon)^* = -1 \quad \text{on all } Q_J^{n-1} \text{ such that } (x_\varepsilon, t_{k_n}) \in \bar{Q}_J^{n-1}.$$

Step 3. There exists $I \in \mathbb{Z}^N$, such that $(x_\varepsilon, t_{k_n}) \in \bar{Q}_I^n$ and $(\theta^\varepsilon)^* = 1$ on Q_I^n.

By contradiction, assume that on all cubes Q_J^n such that $(x_\varepsilon, t_{k_n}) \in \bar{Q}_J^n$, we have $(\theta^\varepsilon)^* = -1$. Then, using *Step 2*, we deduce that $(\theta^\varepsilon)^* = -1$ in a neighborhood of (x_ε, t_{k_n}). This is absurd since $(\theta^\varepsilon)^*(x_\varepsilon, t_{k_n}) = 1$.

Before continuing the proof, we need a few notation. We set

$$\overline{n} = \inf\{k, \quad k_n \le k \le k_{n+1} - 1,\ \theta_I^k = 1\}.$$

In particular, we have $\theta_I^{\overline{n}} = 1$ and $\theta_I^{\overline{n}-1} = -1$.

Since $(\varphi_\varepsilon)_t(x_\varepsilon, t_{k_n}) > 0$ for ε small enough, by Implicit Function Theorem, there exists a neighborhood V_ε of $(x_\varepsilon, t_\varepsilon)$ and a function $\bar{\Psi}_\varepsilon$ such that

$$\{\varphi_\varepsilon(x,t) < 1\} \Leftrightarrow \{t < \bar{\Psi}_\varepsilon(x)\}$$

in V_ε. Using the fact that $(\theta^\varepsilon)^* \le \varphi_\varepsilon$ yields

$$\{(\theta^\varepsilon)^* = 1\} \subset \{t \ge \bar{\Psi}_\varepsilon(x)\}. \tag{5.18}$$

Moreover, for δ_0 small enough, *i.e.* for $(x_\varepsilon, t_\varepsilon)$ closed enough to (x_0, t_0), we can assume that $V_\varepsilon \supset Q_{r,\tau}(x_0, t_0) = B_r(x_0) \times (t_0 - \tau, t_0 + \tau)$. We define $\nu = x_\varepsilon - x_I \in [0, \Delta x)^N$ and $\Psi_\varepsilon(x) = \bar{\Psi}_\varepsilon(x + \nu)$. In particular, we have $\Psi_\varepsilon(x_I) = t_{k_n}$.

Step 4. For all $(x_J, t_{k_m}) \in Q_{r,\tau}(x_0,t_0) = B_r(x_0) \times (t_0 - \tau, t_0 + \tau)$

$$\boldsymbol{\theta^\varepsilon(x_J, t_{k_m}) = 1 \quad \Longrightarrow \quad t_{k_m} \ge \Psi_\varepsilon(x_J).}$$

To prove this, we consider the collection of nodes

$$\mathcal{C} = \{(x_J, t_{k_m}) \in Q_{r,\tau}(x_0, t_0) \cap \{\theta^\varepsilon = 1\}\}.$$

By inclusion (5.18), we then have $t_{k_m} \ge \bar{\Psi}_\varepsilon(x_J)$, $\forall (x_J, t_{k_m}) \in \mathcal{C}$. We deduce $\forall (x_J, t_{k_m}) \in \mathcal{C}$

$$\bar{Q}_J^m = [x_J, x_J + \Delta x] \times [t_{k_m}, t_{k_{m+1}}] \subset \{t \ge \bar{\Psi}_\varepsilon(x)\}.$$

This implies that

$$(x_J + (x_\varepsilon - x_I), t_{k_m}) \in \{t \ge \bar{\Psi}_\varepsilon(x)\}$$

and so

$$(x_J, t_{k_m}) \in \{t \ge \Psi_\varepsilon(x)\}$$

which implies i) because any $t_{m'}$ can be written t_{k_m} for a suitable m.

Step 5. We have $\boldsymbol{\theta_I^m = -1}$ for $\boldsymbol{m_0 \le m \le \overline{n} - 1}$ where $\boldsymbol{m_0 = \inf\{m,\ t_m \ge t_0 - \tau\}}$. By contradiction, suppose that there exists $m_0 \le m \le \overline{n} - 1$ such that $\theta_I^m = 1$. We then define m_1 as

$$m_1 = \sup\{m \le \overline{n} - 1,\ \theta_I^m = 1\}.$$

In particular, we have $\theta_I^{m_1+1} = -1$ (since $\theta_I^{\overline{n}-1} = -1$). Two cases may occur:

(a) $t_{m_1} = t_{k_n} = t_{\overline{n}}$.
In this case, we have $\widehat{c}_I^{m_1} = \widehat{c}_I^{\overline{n}-1} > 0$ (since $\theta_I^{\overline{n}-1} = -1$ and $\theta_I^{\overline{n}} = 1$). This contradicts the fact that $\theta_I^{m_1} = 1$ and $\theta_I^{m_1+1} = -1$.

(b) $t_{m_1} < t_{k_n} = t_{\overline{n}}$.
In this case, we have $\theta^\varepsilon(x_I, t_{m_1}) = 1$ and $t_{m_1} < t_{k_n} = \Psi_\varepsilon(x_I)$. This contradicts *Step 4*.

We now prove *(iii)*.

By Implicit Functions Theorem, we have $\varphi_\varepsilon(x, \bar{\Psi}_\varepsilon(x)) = 1$. Deriving yields

$$\varphi_t(x, \bar{\Psi}_\varepsilon(x))D\bar{\Psi}_\varepsilon(x) + D\varphi(x, \bar{\Psi}(x)) = 0.$$

Taking $x = x_\varepsilon$ yields

$$D\Psi_\varepsilon(x_I) = -\frac{D\varphi(x_\varepsilon, \bar{\Psi}_\varepsilon(x_\varepsilon))}{\varphi_t(x_\varepsilon, \bar{\Psi}_\varepsilon(x_\varepsilon))} = -\frac{D\varphi(x_I, t_{k_n})}{\varphi_t(x_I, t_{k_n})} + O(\Delta x)$$

and so

$$D\Psi_\varepsilon(x_I) = -\frac{D\varphi(x_0, t_0)}{\varphi_t(x_0, t_0)} + O(|x_I - x_0| + |t_{k_n} - t_0| + \Delta x).$$

Moreover, by Taylor expansion, we get, if $|\varphi(x_\varepsilon, t_\varepsilon) - 1|$ is small enough, for all $J \in V(I)$

$$\begin{aligned}\Psi_\varepsilon(x_J) =& \Psi_\varepsilon(x_I) + (x_J - x_I)\cdot D\Psi_\varepsilon(x_I) + O(|\Delta x|^2)\\ =& \Psi_\varepsilon(x_I) - \frac{D\varphi(x_0, t_0)}{\varphi_t(x_0, t_0)}\cdot(x_J - x_I) + (\Delta x)\ O\left(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|\right).\end{aligned}$$

where "the O is uniform in ε". This ends the proof of (iii).

It just remains to show that if $\varphi_t(x_0, t_0) < 0$, then for all $(x_J, t_{k_n}) \in Q_{r,\tau}(x_0, t_0) = B_r(x_0) \times (t_0 - \tau, t_0 + \tau)$

$$\theta^\varepsilon(x_J, t_{k_{n-1}}) = 1 \quad \text{and} \quad \theta^\varepsilon(x_J, t_{k_n}) = -1 \quad \Longrightarrow \quad t_{k_n} \le \Psi_\varepsilon(x_J).$$

In this case, inclusion (5.18) is replaced by

$$\{(\theta^\varepsilon)^* = 1\} \subset \{t \le \bar{\Psi}_\varepsilon(x)\}.$$

By definition of θ^ε, for all $y \in [x_J, x_J + \Delta x]$, we have $(\theta^\varepsilon)^*(y, t_{k_n}) = 1$. Taking $y = x_J + \nu$, we then deduce that

$$t_{k_n} \le \bar{\Psi}_\varepsilon(y) = \bar{\Psi}_\varepsilon(x_J + \nu) = \Psi_\varepsilon(x_J).$$

$\square$

Lemma 5.4 (Approximate horizontal level set in the i-direction for negative velocity)
Under the notation and assumptions of Lemma 5.3 with $\varphi_t(x_0, t_0) < 0$, let us suppose that there exists $\delta_0 > 0$ such that $c < -\delta < 0$ on $B_{\delta_0}(x_0, t_0)$.
Let us assume moreover that $(x_I, t_{\overline{n}}) \in B_{\delta_0}(x_0, t_0)$, $\theta_I^{\overline{n}-1} = 1$ and $\theta_I^{\overline{n}} = -1$. If for some fixed $i \in \{1, ..., N\}$ we have

$$\tilde{u}_I^{\overline{n}-1} - \widehat{u}_{-,I^{i,+}}^{\overline{n}-1} < 0 \quad \text{and} \quad \tilde{u}_I^{\overline{n}-1} - \widehat{u}_{-,I^{i,-}}^{\overline{n}-1} < 0$$

then

$$\left|\frac{D\varphi(x_0, t_0)}{\varphi_t(x_0, t_0)}\cdot e_i\right| \le o(1).$$

Proof
We first prove that if $\tilde{u}_I^{\overline{n}-1} - \widehat{u}_{-,J}^{\overline{n}-1} < 0$ for some $J \in V(I)\setminus\{I\}$, then

$$\Psi_\varepsilon(x_I) - \Psi_\varepsilon(x_J) \leq o(\Delta x).$$

There are two cases: $\widehat{u}_{-,J}^{\overline{n}-1} = \infty$ or $\widehat{u}_{-,J}^{\overline{n}-1} < \infty$.
If $\widehat{u}_{-,J}^{\overline{n}-1} < \infty$ then $J \in F_-^{\overline{n}-1}$. By Lemma 5.1 it results

$$\widehat{u}_{-,J}^{\overline{n}-1} = \sup\{t_m \leq t_{\overline{n}-1},\ \theta_J^{m-1} = 1, \theta_J^p = -1, \quad \text{for} \quad m \leq p \leq \overline{n}-1\}$$

and by Lemma 5.3 (iv) we have $\widehat{u}_{-,J}^{\overline{n}-1} \leq \Psi_\varepsilon(x_J)$.
We then deduce that

$$0 > \tilde{u}_I^{\overline{n}-1} - \widehat{u}_{-,J}^{\overline{n}-1} \geq \tilde{t}_{\overline{n}} - \Psi_\varepsilon(x_J) = \Psi_\varepsilon(x_I) - \Psi_\varepsilon(x_J) - (t_{\overline{n}} - \tilde{t}_{\overline{n}}).$$

We apply Lemma 5.2 and we obtain

$$\Psi_\varepsilon(x_I) - \Psi_\varepsilon(x_J) \leq o(\Delta x).$$

If $\widehat{u}_{-,J}^{\overline{n}-1} = \infty$ then necessarily $\theta_J^{\overline{n}-1} = 1$, now either $\theta_J^{\overline{n}} = 1$, respectively either $\theta_J^{\overline{n}} = -1$. Then we can apply Lemma 5.3 (i), respectively (iv), and we get $t_{\overline{n}} \leq \Psi_\varepsilon(x_J)$. We deduce then

$$\Psi_\varepsilon(x_I) - \Psi_\varepsilon(x_J) \leq t_{\overline{n}} - t_{\overline{n}} \leq 0.$$

Using Lemma 5.3 (iii) for $J = I^{i,\pm}$, we deduce that

$$\pm\Delta x \frac{D\varphi(x_0,t_0)}{\varphi_t(x_0,t_0)} \cdot e_i \leq o(\Delta x).$$

□

Lemma 5.5 (Decay of θ^ε in the gradient direction of a test function)
Let φ be C^2 in a neighborhood V of (x_0,t_0) and let us suppose there exist $\delta_0 > 0$ such that $\max_{\overline{V}}((\theta^\varepsilon)^* - \varphi) = (\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon) - \varphi(x_\varepsilon,t_\varepsilon)$ with $(x_\varepsilon,t_\varepsilon) \in B_{\delta_0}(x_0,t_0) \subset V$ and $(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon) = 1$. Then, there exists a node $(I,n) \in \mathbb{Z}^N \times \mathbb{N}$ such that $\theta_I^{k_{n+1}-1} = 1$ with $(x_\varepsilon,t_\varepsilon) \in \partial Q_I^n = \partial(]x_I, x_I + \Delta x[\times]t_{k_n}, t_{k_{n+1}}[)$ such that if $\mp e_i \cdot D\varphi(x_0,t_0) > 0$ then

$$\theta^\varepsilon(x,t) = -1 \quad \text{in } Q_{I^{i,\pm}}^n =]x_{I^{i,\pm}}, x_{I^{i,\pm}} + \Delta x[\times]t_{k_n}, t_{k_{n+1}}[.$$

Proof of Lemma 5.5
Since $(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon) = 1$, there exists a node $(I,n) \in \mathbb{Z}^N \times \mathbb{N}$ such that $\theta_I^{k_{n+1}-1} = 1$ with $(x_\varepsilon,t_\varepsilon) \in \partial Q_I^n = \partial(]x_I, x_I + \Delta x[\times]t_{k_n}, t_{k_{n+1}}[)$.
Assume for example that

$$e_i \cdot D\varphi(x_0,t_0) < 0$$

and let us suppose by contradiction that $\theta^\varepsilon = 1$ in $Q_{I^{i,+}}^n =]x_{I^{i,+}}, x_{I^{i,+}} + \Delta x[\times]t_{k_n}, t_{k_{n+1}}[$.
We define $\varphi_\varepsilon = \varphi + ((\theta^\varepsilon)^* - \varphi)(x_\varepsilon,t_\varepsilon)$. In particular, we have $(\theta^\varepsilon)^* \leq \varphi_\varepsilon$ and $(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon) = \varphi_\varepsilon(x_\varepsilon,t_\varepsilon) = 1$. Since $(\theta^\varepsilon)^* \leq \varphi_\varepsilon$, the following inclusion holds

$$\{(\theta^\varepsilon)^* = 1\} \subset \{\varphi_\varepsilon \geq 1\}.$$

We define $x_\varepsilon^{i,\lambda} = x_\varepsilon + \lambda e_i$ with $0 \le \lambda \le \Delta x$ such that $(\theta^\varepsilon)^*(x_\varepsilon^{i,\lambda}, t_\varepsilon) = 1$. Then $\varphi_\varepsilon(x_\varepsilon^{i,\lambda}, t_\varepsilon) \ge 1$ and

$$\frac{\varphi_\varepsilon(x_\varepsilon^{i,\lambda}, t_\varepsilon) - \varphi_\varepsilon(x_\varepsilon, t_\varepsilon)}{\lambda} \ge 0.$$

Taking the limit for $\lambda \to 0$, we obtain

$$e_i \cdot D\varphi(x_\varepsilon, t_\varepsilon) = e_i \cdot D\varphi_\varepsilon(x_\varepsilon, t_\varepsilon) \ge 0.$$

This ends the proof, since it contradicts the assumption. □

Lemma 5.6 (Bound on $|t_\varepsilon - t_{\overline{m}_0}|$ for negative velocity)
Under the notation and assumptions of Lemma 5.5, if we suppose there exists $\delta > 0$ and $\delta_0 > 0$ such that $c(x,t) < -\delta < 0$ in $(x,t) \in B_{2\delta_0}(x_0,t_0) \subset V$ then the following estimate holds

$$|t_\varepsilon - t_{\overline{m}_0}| \le \frac{\Delta x}{\delta}$$

with

$$t_{\overline{m}_0} = \sup\{t_m \le t_{k_n} \ : \ \theta_{I^{i,+}}^{m-1} = 1,\ \theta_{I^{i,+}}^{m} = -1\} \quad \textit{if we assume} \quad - e_i \cdot D\varphi(x_0,t_0) > 0$$

$$(\textit{resp.} \quad t_{\overline{m}_0} = \sup\{t_m \le t_{k_n} \ : \ \theta_{I^{i,-}}^{m-1} = 1,\ \theta_{I^{i,-}}^{m} = -1\} \quad \textit{if we assume} \quad + e_i \cdot D\varphi(x_0,t_0) > 0)$$

where I is defined in Lemma 5.5.

Proof of Lemma 5.6
Let us define

$$\overline{m}_0 = \sup\{\ m \le k_{n+1} - 1,\ \theta_{I^{i,\pm}}^{m-1} = 1, \theta_{I^{i,\pm}}^{m} = -1\}.$$

For $\Delta x, \Delta t$ small enough, we can assume that $(x_K, t_m) \in B_{2\delta_0}(x_0,t_0)$ for $K = I, I^{i,\pm}$ and $\overline{m}_0 \le m \le k_{n+1}$. Since $c < 0$ in $B_{2\delta_0}(x_0,t_0)$, $\theta_I^{k_{n+1}-1} = 1$ implies $\theta_I^m = 1$ for all $\overline{m}_0 \le m \le k_{n+1} - 1$, and by definition of $\overline{m}_0$, $\theta_{I^{i,\pm}}^m = -1$ for all $\overline{m}_0 \le m \le k_{n+1} - 1$.
This means that $I^{i,\pm} \in F_-^m$ for all $\overline{m}_0 \le m \le k_{n+1} - 1$ and so

$$\widehat{u}_{-,I^{i,\pm}}^m = t_{\overline{m}_0} \text{ for } \overline{m}_0 \le m \le k_{n+1} - 1. \tag{5.19}$$

In particular, $\widehat{u}_{-,I^{i,\pm}}^{k_{n+1}-1} = t_{\overline{m}_0}$ and by the definition of the $\widehat{t}_{k_{n+1}}$ it results $\tilde{u}_I^{k_{n+1}-1} \ge \widehat{t}_{k_{n+1}}$ with $\widehat{t}_{k_{n+1}} = t_{k_{n+1}}$, since $t_{k_{n+1}} > t_{k_n}$.
By the equation

$$\sum_{k=1}^{N} \left(\max_{\pm} \left(0, \tilde{u}_I^{k_{n+1}-1} - \widehat{u}_{-,I^{k,\pm}}^{k_{n+1}-1}\right)\right)^2 = \left(\frac{\Delta x}{\widehat{c}_I^{k_{n+1}-1}}\right)^2,$$

we conclude that

$$t_\varepsilon - t_{\overline{m}_0} \le t_{k_{n+1}} - t_{\overline{m}_0} \le \tilde{u}_I^{k_{n+1}-1} - \widehat{u}_{-,I^{i,\pm}}^{k_{n+1}-1} \le \frac{(\Delta x)}{|\widehat{c}_I^{k_{n+1}-1}|} \le \frac{\Delta x}{\delta}.$$

□

6 Proof of Theorem 2.5

This section is dedicated to the proof of the main theorem, which is preceded by two important propositions.

The first proposition will show that the limit function $\overline{\theta}^0$ is a sub-solution in all the domain excepted for the initial time, whereas the second proposition will show that the limit function $\overline{\theta}^0$ is a sub-solution at the initial time. The reason why we need to treat a part the initial condition is that the proof of the first proposition is based on the definition of discontinuous viscosity sub-solution (see Barles [1] and Crandall, Ishii, Lions [6]) consisting in testing the equation by smooth functions, but this definition does not hold at the initial time. Then we treat the initial condition using the technique of barriers.

At the end of this section, we give the main proof using both results.

Proposition 6.1 (Sub-solution property of the limit)
The function $\overline{\theta}^0$ is a sub-solution of the equation

$$\theta_t(x,t) = c(x,t)|D\theta(x,t)|$$

on $\mathbb{R}^N \times (0,T)$.

Proof of Proposition 6.1
By contradiction, assume that there are (x_0,t_0) and $\varphi \in C^2$ such that $\overline{\theta}^0 - \varphi$ reaches a strict maximum at (x_0,t_0) with $\overline{\theta}^0(x_0,t_0) = \varphi(x_0,t_0)$ and

$$\varphi_t(x_0,t_0) = \alpha + c(x_0,t_0)|D\varphi(x_0,t_0)| \tag{6.20}$$

with $\alpha > 0$. Since the maximum of $\overline{\theta}^0 - \varphi$ is strict, there exists $(x_\varepsilon,t_\varepsilon) \to (x_0,t_0)$ as $\Delta x \to 0$ such that

$$\max((\theta^\varepsilon)^* - \varphi) = ((\theta^\varepsilon)^* - \varphi)(x_\varepsilon,t_\varepsilon).$$

In particular, we have $(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon) = 1$ for $\Delta x, \Delta t$ small enough. Indeed, by contradiction, suppose that $(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon) = -1$. Using the fact that $(\theta^\varepsilon)^*$ is upper semi-continuous, we obtain $(\theta^\varepsilon)^* = -1$ a neighborhood of $(x_\varepsilon,t_\varepsilon)$. We then deduce that $\varphi_t(x_\varepsilon,t_\varepsilon) = D\varphi(x_\varepsilon,t_\varepsilon) = 0$ and so

$$0 = \varphi_t(x_\varepsilon,t_\varepsilon) - c(x_\varepsilon,t_\varepsilon)|D\varphi(x_\varepsilon,t_\varepsilon)| \to \varphi_t(x_0,t_0) - c(x_0,t_0)|D\varphi(x_0,t_0)| = \alpha$$

This is absurd.
If $|D\varphi(x_0,t_0)| \neq 0$, we note that we can rewrite inequality (6.20) as

$$\varphi_t(x_0,t_0) = \bar{c}|D\varphi(x_0,t_0)| \quad \text{with } \bar{c} > c(x_0,t_0) \tag{6.21}$$

We denote by

$$\vec{n}_0 = \frac{D\varphi(x_0,t_0)}{|D\varphi(x_0,t_0)|}. \tag{6.22}$$

To continue the proof, we have to distinguish several cases:

1. $\boldsymbol{c(x_0,t_0)>0}$.
   In this case, we have in particular, $\varphi_t(x_0,t_0)>0$. Then we can apply Lemma 5.3 and we deduce that there exist $\Psi_\varepsilon\in C^2$ and $(I,n)\in\mathbb{Z}^N\times\mathbb{N}$ such that $(x_I,t_{k_n})\to(x_0,t_0)$ as $\varepsilon=(\Delta x,\Delta t)\to 0$,
$$(\theta^\varepsilon)^*(x_I,t_{k_n})=1,\quad t_{k_n}=\Psi_\varepsilon(x_I)$$
   and
$$\theta_I^{\overline{n}}=1,\quad \theta_I^{\overline{n}-1}=-1,$$
   where $\bar{n}$ is defined in Lemma 5.3. Using Lemma 5.1 and Lemma 5.3 (i), we deduce also that for all $J\in V(I)\backslash\{I\}$ such that $\theta_J^{\overline{n}-1}=1$, we have
$$\widehat{u}_{+,J}^{\overline{n}-1}\geq\Psi_\varepsilon(x_J).$$
   This implies for all $J\in V(I)\cap F_+^{\overline{n}-1}$, using also the (general) fact that $\tilde{u}_I^{\overline{n}-1}\leq t_{\overline{n}}=t_{k_n}$,
$$\tilde{u}_I^{\overline{n}-1}-\widehat{u}_{+,J}^{\overline{n}-1}\leq t_{\overline{n}}-\widehat{u}_{+,J}^{\overline{n}-1}=\Psi_\varepsilon(x_I)-\widehat{u}_{+,J}^{\overline{n}-1}\leq\Psi_\varepsilon(x_I)-\Psi_\varepsilon(x_J). \tag{6.23}$$
   By the GFMM algorithm (Step 5), $\tilde{u}_I^{\overline{n}-1}$ is solution of the equation
$$\left(\frac{\Delta x}{c(x_I,t_{\overline{n}-1})}\right)^2=\sum_{i=1}^N\left(\max_{\pm}\left(0,\tilde{u}_I^{\overline{n}-1}-\widehat{u}_{+,I^{i,\pm}}^{\overline{n}-1}\right)\right)^2$$
   If $|D\varphi(x_0,t_0)|\neq 0$, by adding (6.23) for $J=I^{i,\pm}$ on all direction $i\in\mathcal{C}\subset\{1,..,N\}$ such that
$$\tilde{u}_I^{\overline{n}-1}-\widehat{u}_{+,I^{i,+}}^{\overline{n}-1}\geq 0\ \text{ or }\ \tilde{u}_I^{\overline{n}-1}-\widehat{u}_{+,I^{i,-}}^{\overline{n}-1}\geq 0$$
   and by using Lemma 5.3 *(iii)*, we can estimate
$$\begin{aligned}\left(\frac{\Delta x}{c(x_I,t_{\overline{n}-1})}\right)^2&=\sum_{i\in\mathcal{C}}\left(\max_{\pm}\left(\tilde{u}_I^{\overline{n}-1}-\widehat{u}_{+,I^{i,\pm}}^{\overline{n}-1}\right)\right)^2\leq\sum_{i\in\mathcal{C}}\max_{\pm}\left(\Psi_\varepsilon(x_I)-\Psi_\varepsilon(x_{I^{i,\pm}})\right)^2\\&\leq\frac{(\Delta x)^2}{\bar{c}^2}\sum_{i\in\mathcal{C}}(\vec{n_0}\cdot e_i)^2+(\Delta x)^2O(\Delta x+|x_I-x_0|+|t_{k_n}-t_0|)\\&\leq\frac{(\Delta x)^2}{\bar{c}^2}+(\Delta x)^2O(\Delta x+|x_I-x_0|+|t_{k_n}-t_0|)\end{aligned}$$
   where $\bar{c}$ and $\vec{n}_0$ are defined in (6.21) and (6.22) respectively.
   It follows that
$$\frac{1}{c^2(x_I,t_{\overline{n}-1})}-\frac{1}{\bar{c}^2}\leq O(\Delta x+|x_I-x_0|+|t_{k_n}-t_0|).$$
   Taking the limit $\varepsilon=(\Delta x,\Delta t)\to 0$, we obtain a contradiction.
   If $D\varphi(x_0,t_0)=0$, we get in the same way
$$\frac{1}{c^2(x_I,t_{\overline{n}-1})}\leq O(\Delta x+|x_I-x_0|+|t_{k_n}-t_0|).$$
   Taking the limit $\varepsilon\to 0$, since we have assumed $c(x_0,t_0)>0$, we obtain a contradiction.

2. $\boldsymbol{c(x_0,t_0)<0}$.
   In this case, we have no informations on the sign of φ_t, so we have to distinguish several cases:

   (a) $\boldsymbol{\varphi_t(x_0,t_0)<0}$.
   Note that, in this case, $|D\varphi(x_0,t_0)|\neq 0$ and (6.21) holds with $0>\bar c>c(x_0,t_0)$. Then we can apply Lemma 5.3 and we deduce that there exist $\Psi_\varepsilon\in C^2$ and $(I,n)\in\mathbb{Z}^N\times\mathbb{N}$ such that

$$(\theta^\varepsilon)^*(x_I,t_{k_n})=1,\quad t_{k_n}=\Psi_\varepsilon(x_I)$$

   and

$$\theta_I^{\overline{n}}=-1,\quad \theta_I^{\overline{n}-1}=1,$$

   where $\bar n$ is defined in Lemma 5.3. Using Lemma 5.1 and Lemma 5.3 (iv), we deduce also that for all $J\in V(I)\backslash\{I\}$ such that $\theta_J^{\overline{n}-1}=-1$, we have

$$\widehat{u}_{-,J}^{\overline{n}-1}\leq\Psi_\varepsilon(x_J).$$

   This implies that for all $J\in V(I)\cap F_-^{\overline{n}-1}$

$$\tilde u_I^{\overline{n}-1}-\widehat{u}_{-,J}^{\overline{n}-1}\geq\tilde t_{\overline{n}}-\Psi_\varepsilon(x_J)=t_{\overline{n}}-\Psi_\varepsilon(x_J)+(\tilde t_{\overline{n}}-t_{\overline{n}})=\Psi_\varepsilon(x_I)-\Psi_\varepsilon(x_J)+(\tilde t_{\overline{n}}-t_{\overline{n}})$$

   Since $c(x_0,t_0)\neq 0$, there exists $\delta,\ \delta_0>0$ such that $|c|\geq\delta>0$ on $B_{\delta_0}(x_0,t_0)$ and we can apply Lemma 5.2 to get

$$\tilde u_I^{\overline{n}-1}-\widehat{u}_{-,J}^{\overline{n}-1}\geq\Psi_\varepsilon(x_I)-\Psi_\varepsilon(x_J)+o(\Delta x).$$

   Using Lemma 5.3 (iii) yields

$$\tilde u_I^{\overline{n}-1}-\widehat{u}_{-,J}^{\overline{n}-1}\geq\frac{1}{\bar c}\vec{n_0}\cdot(x_J-x_I)+(\Delta x)\ O(\Delta x+|x_I-x_0|+|t_{k_n}-t_0|)+o(\Delta x).$$

   By adding the previous equation for $J=I^{i,\pm}$ on all direction $i\in\mathcal{C}\subset\{1,..,N\}$ such that

$$\tilde u_I^{\overline{n}-1}-\widehat{u}_{-,I^{i,+}}^{\overline{n}-1}\geq 0\ \text{ or }\ \tilde u_I^{\overline{n}-1}-\widehat{u}_{-,I^{i,-}}^{\overline{n}-1}\geq 0$$

   we obtain, since $|D\varphi(x_0,t_0)|\neq 0$

$$\begin{aligned}\left(\frac{\Delta x}{c(x_I,t_{\overline{n}-1})}\right)^2&=\sum_{i=1}^N\left(\max_{\pm}\left(0,\tilde u_I^{\overline{n}-1}-\widehat{u}_{-,I^{i,\pm}}^{\overline{n}-1}\right)\right)^2\\&=\sum_{i\in\mathcal{C}}\left(\tilde u_I^{\overline{n}-1}-\widehat{u}_{-,I^{i,\pm}}^{\overline{n}-1}\right)^2\\&\geq\frac{(\Delta x)^2}{\bar c^2}\sum_{i\in\mathcal{C}}(\vec{n_0}\cdot e_i)^2+(\Delta x)^2\ O(\Delta x+|x_I-x_0|+|t_{k_n}-t_0|)+o(\Delta x)^2\end{aligned}\tag{6.24}$$

   If $i\notin\mathcal{C}$ (*i.e.* $\tilde u_I^{\overline{n}-1}-\widehat{u}_{-,I^{i,+}}^{\overline{n}-1}<0$ and $\tilde u_I^{\overline{n}-1}-\widehat{u}_{-,I^{i,-}}^{\overline{n}-1}<0$), then by Lemma 5.4, we deduce that

$$\left|\frac{1}{\bar c}\Delta x\ \vec{n_0}\cdot e_i\right|=o(\Delta x).\tag{6.25}$$

By combining (6.24) and (6.25), we get

$$\begin{aligned}\left(\frac{\Delta x}{c(x_I,t_{\overline{n}-1})}\right)^2 \geq & \frac{(\Delta x)^2}{\bar{c}^2}\sum_{i=1}^{N}(\vec{n_0}\cdot e_i)^2 + (\Delta x)^2\ O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|) + o(\Delta x^2)\\ = & \frac{(\Delta x)^2}{\bar{c}^2} + (\Delta x)^2\ O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|) + o(\Delta x^2)\end{aligned}$$

This implies

$$\frac{1}{c^2(x_I,t_{\overline{n}-1})} - \frac{1}{\bar{c}^2} \geq O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|) + o(1).$$

Taking the limit $\varepsilon = (\Delta x, \Delta t) \to 0$, we get the contradiction since $|c(x_0,t_0)| > |\bar{c}|$.

(b) $\boldsymbol{\varphi_t(x_0,t_0) > 0}$.

Since $c(x_0,t_0) < 0$, we have by the algorithm that $\dfrac{\partial(\theta^\varepsilon)^*}{\partial t} \leq 0$. We define $\varphi_\varepsilon = \varphi + ((\theta^\varepsilon)^* - \varphi)(x_\varepsilon,t_\varepsilon)$. In particular, we have $(\theta^\varepsilon)^* \leq \varphi_\varepsilon$ and

$$(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon) = \varphi_\varepsilon(x_\varepsilon,t_\varepsilon) = 1.$$

We have $t_\varepsilon = t_{k_n}$. Indeed, assume that $t_\varepsilon \in (t_{k_n}, t_{k_{n+1}})$. Using the fact that $(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon) = 1$, we deduce that $(\theta^\varepsilon)^*(x_\varepsilon,t) = 1$ for $t_{k_n} \leq t \leq t_{k_{n+1}}$ and so $\varphi_t(x_\varepsilon,t_\varepsilon) = 0$. This is absurd for ε small enough since $\varphi_t(x_0,t_0) > 0$.

Using the fact that $(\varphi_\varepsilon)_t > 0$, we deduce that $(\theta^\varepsilon)^*(x_\varepsilon,t) \leq \varphi_\varepsilon(x_\varepsilon,t) < 1$ for $t < t_{k_n}$. This is absurd since $\dfrac{\partial(\theta^\varepsilon)^*}{\partial t} \leq 0$.

(c) $\boldsymbol{\varphi_t(x_0,t_0) = 0}$.

Since the equation (6.20) holds with $\alpha > 0$, we have, in particular, $|D\varphi(x_0,t_0)| \neq 0$. Then, there exists a direction $\pm e_i$ such that $\mp e_i \cdot D\varphi(x_0,t_0) > 0$. Using Lemma 5.5, we deduce that there exists $(I,n) \in \mathbb{Z}^N \times \mathbb{N}$ such that $\theta_I^{k_{n+1}-1} = 1$ and $\theta^\varepsilon = -1$ on $Q^n_{I^{i,\pm}} =]x_{I^{i,\pm}}, x_{I^{i,\pm}} + \Delta x[\times]t_{k_n}, t_{k_{n+1}}[$. We define $t_{\overline{m}_0}$ such that

$$\overline{m}_0 = \sup\{m\ :\ t_m \leq t_{k_n},\ \theta^{m-1}_{I^{i,\pm}} = 1,\ \theta^m_{I^{i,\pm}} = -1\}.$$

In particular, $(\theta^\varepsilon)^*(x,t_{\overline{m}_0}) = 1$ for all $x \in [x_{I^{i,\pm}}, x_{I^{i,\pm}} + \Delta x]$.

We define $\varphi_\varepsilon = \varphi + ((\theta^\varepsilon)^* - \varphi)(x_\varepsilon,t_\varepsilon)$. In particular, we have $(\theta^\varepsilon)^* \leq \varphi_\varepsilon$ and

$$(\theta^\varepsilon)^*(x_\varepsilon,t_\varepsilon) = \varphi_\varepsilon(x_\varepsilon,t_\varepsilon) = 1.$$

Since the following inclusion $\{(\theta^\varepsilon)^* = 1\} \subset \{\varphi_\varepsilon \geq 1\}$ holds, $\varphi_\varepsilon(x,t_{\overline{m}_0}) \geq 1$ for all $x \in [x_{I^{i,\pm}}, x_{I^{i,\pm}} + \Delta x]$.

Let $\nu \in [0,\Delta x]^N$ be such that $x_\varepsilon = x_I + \nu$ and let us define $y \equiv x_{I^{i,\pm}} + \nu$ and $\overline{\varphi}(\cdot,\cdot) \equiv \varphi_\varepsilon(\cdot+\nu,\cdot)$. Then it yields $\overline{\varphi}(x_I,t_\varepsilon) = \varphi_\varepsilon(x_\varepsilon,t_\varepsilon) = 1$, and $\overline{\varphi}(x_{I^{i,\pm}},t_{\overline{m}_0}) = \varphi_\varepsilon(y,t_{\overline{m}_0}) \geq 1$.

To obtain the contradiction, we consider the expansion of $\overline{\varphi}$ up to the first order

$$\begin{aligned}0\ & \leq \overline{\varphi}(x_{I^{i,\pm}},t_{\overline{m}_0}) - \overline{\varphi}(x_I,t_\varepsilon)\\ & \leq (x_{I^{i,\pm}} - x_I)\cdot D\overline{\varphi}(x_I,t_\varepsilon) + (t_{\overline{m}_0} - t_\varepsilon)\partial_t\overline{\varphi}(x_I,t_\varepsilon) + O((\Delta x)^2 + |t_\varepsilon - t_{\overline{m}_0}|^2).\end{aligned}$$

Now by Lemma 5.6 and using the fact that $\partial_t\varphi(x_0,t_0)=0$ we obtain

$$\pm e_i\cdot D\overline{\varphi}(x_I,t_\varepsilon)\Delta x+o(\Delta x)\geq 0,$$

that is absurd, since by assumption $\pm e_i\cdot D\varphi(x_0,t_0)<0$.

3. $\boldsymbol{c(x_0,t_0)=0.}$
In this case, we have

$$\varphi_t=\alpha>0$$

and we can apply Lemma 5.3. Hence, there exists $r,\ \tau>0$, a function $\Psi_\varepsilon\in C^2(B_r(x_0),(t_0-\tau,t_0+\tau))$ and a node $(I,n)\in\mathbb{Z}^N\times\mathbb{N}$ such that

$$(\theta^\varepsilon)^*(x_I,t_{k_n})=1,\quad t_{k_n}=\Psi_\varepsilon(x_I)$$

and for all $J\in V(I),\ t_m\in(t_0-\tau,t_0+\tau)$, we have

$$\theta^\varepsilon(x_J,t_m)=1\implies t_m\geq\Psi_\varepsilon(x_J)\tag{6.26}$$

We define m_0 such that

$$t_{m_0-1}<t_0-\tau\leq t_{m_0}.$$

For all $J\in(V(I)\setminus\{I\})\cap\{\theta^{\overline{n}-1}=1\}$ (with $\bar{n}$ defined in Lemma 5.3), we define

$$m_J=\sup\{k\leq\overline{n},\ \theta_J^{k-1}=-1\}$$

We distinguish two cases:

(a) There exists $J\in(V(I)\setminus\{I\})\cap\{\theta^{\overline{n}-1}=1\}$ such that $m_J<m_0$.

Using the fact that $\theta_I^k=-1$ for $m_0\leq k\leq\overline{n}-1$ (see Lemma 5.3 *(ii)*), we have that $J\in F_+^k,\ \forall\, m_0\leq k\leq\overline{n}-1$ and we deduce that

$$\widehat{u}_{+,J}^{\overline{n}-1}=u_J^{\overline{n}-1}\leq t_{m_0}\quad\text{and}\quad\theta^\varepsilon(x_J,t_{m_0})=1.$$

By (6.26), we then have $t_{m_0}\geq\Psi_\varepsilon(x_J)$.
We now assume that $|D\varphi|\neq 0$ (the case $|D\varphi|=0$ can be treated in a similar way). Using Lemma 5.3, we deduce that

$$t_{m_0}\geq\Psi_\varepsilon(x_J)=t_{k_n}-\frac{1}{\bar{c}}\,\vec{n}_0\cdot(x_J-x_I)+(\Delta x)\ O(\Delta x+|x_I-x_0|+|t_{k_n}-t_0|),$$

and so

$$\begin{aligned}t_{k_n}-t_{m_0}\leq&\frac{1}{\bar{c}}\,\vec{n}_0\cdot(x_J-x_I)+(\Delta x)\ O(\Delta x+|x_I-x_0|+|t_{k_n}-t_0|)\\ \leq&\frac{\Delta x}{\bar{c}}+(\Delta x)\ O(\Delta x+|x_I-x_0|+|t_{k_n}-t_0|).\end{aligned}$$

Sending $\Delta x,\Delta t$ to 0, yields

$$t_0-(t_0-\tau)=\tau\leq 0.$$

This is absurd.

(b) For all $J \in (V(I) \setminus \{I\}) \cap \{\theta^{\overline{n}-1} = 1\}$, $m_J \geq m_0$.
We then have $\theta^\varepsilon(x_J, t_{m_J}) = 1$ and so by (6.26) we have $\widehat{u}^{\overline{n}-1}_{+,J} = t_{m_J} \geq \Psi(x_J)$.
We now assume that $|D\varphi| \neq 0$ (the case $|D\varphi| = 0$ can be treated in a similar way). Using Lemma 5.3, we deduce that

$$\widehat{u}^{\overline{n}-1}_{+,J} \geq \Psi(x_J) = t_{k_n} - \frac{1}{\bar{c}}\vec{n}_0 \cdot (x_J - x_I) + (\Delta x)\ O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|),$$

and so

$$\tilde{u}^{\overline{n}-1}_I - \widehat{u}^{\overline{n}-1}_{+,J} \leq t_{k_n} - \widehat{u}^{\overline{n}-1}_{+,J} \leq \frac{1}{\bar{c}}\vec{n}_0 \cdot (x_J - x_I) + (\Delta x)\ O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|)$$

By adding for $J = I^{i,\pm}$ on all directions $i \in \mathcal{C} \subset \{1, .., N\}$ such that

$$\tilde{u}^{\overline{n}-1}_I - \widehat{u}^{\overline{n}-1}_{+,I^i} = \max(\tilde{u}^{\overline{n}-1}_I - \widehat{u}^{\overline{n}-1}_{+,I^{i,+}}, \tilde{u}^{\overline{n}-1}_I - \widehat{u}^{\overline{n}-1}_{+,I^{i,-}}) \geq 0,$$

we deduce that

$$\begin{aligned}\left(\frac{\Delta x}{\widehat{c}^{\overline{n}-1}_I}\right)^2 &= \sum_{i \in \mathcal{C}} \left(\tilde{u}^n_I - \widehat{u}^{n-1}_{+,I^i}\right)^2 \\ &\leq \left(\frac{\Delta x}{\bar{c}}\right)^2 + (\Delta x)^2\ O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|).\end{aligned}$$

i.e.

$$\frac{1}{|\widehat{c}^{\overline{n}-1}_I|^2} \leq \frac{1}{\bar{c}^2} + O(\Delta x + |x_I - x_0| + |t_{k_n} - t_0|)$$

Sending $\Delta x, \Delta t$ to 0, yields a contradiction since $\bar{c} > c(x_0, t_0) = 0$. □

We construct a barrier sub-solution and we prove that $\overline{\theta}^0$ defined by (2.8) satisfies the initial condition of (1.1):

Proposition 6.2 (Initial condition)
We have the following inequality:

$$\overline{\theta}^0(\cdot, 0) \leq \left(1_{\Omega_0} - 1_{\Omega_0^c}\right)^*. \tag{6.27}$$

Proof of Proposition 6.2
For $\alpha > 0$ which will be precised later, we consider the following function

$$v(x) = \alpha \ \mathrm{dist}(x, \Omega_0). \tag{6.28}$$

and we define, for all $I \in \mathbb{Z}^N$

$$v_I = v(x_I).$$

We then define for $x_I \in \Omega_0^c$ a velocity $\infty > c_{v,I} > 0$ by solving

$$\sum_{k=1}^{N} (\max_{\pm}(0, v_I - \widehat{v}_{I^{k,\pm}}))^2 = \left(\frac{\Delta x}{c_{v,I}}\right)^2,$$

where

$$\widehat{v}_J = \begin{cases} v_J & \text{if} \quad v_J \le v_I \\ \infty & \text{if} \quad v_J > v_I. \end{cases}$$

This define a GFMM with velocity $c_{v,I}$ and whose solution is v_I. On the one hand, using the fact that $|v_I - v_J| \le \alpha \Delta x$, yields for $J \in V(I)$

$$c_{v,I} \ge \frac{1}{\alpha\sqrt{N}} \tag{6.29}$$

On the other hand, the C^2 regularity of $\partial\Omega_0$ implies that $c_{v,I}$ is uniformly bounded as $\Delta x \to 0$ in a neighborhood of $\partial\Omega_0$.

Moreover, we can define θ_v^ε in the following way

$$\theta_v^\varepsilon(x,t) = \begin{cases} 1 & \text{if} \quad x \in [x_I, x_I + \Delta x[\text{ and } t \ge v_I \\ -1 & \text{if} \quad x \in [x_I, x_I + \Delta x[\text{ and } t < v_I. \end{cases}$$

We denote by u the solution of the GFMM algorithm with velocity $c(x,t)$. We then have

$$\theta^0_{u,I} = 1 \Rightarrow x_I \in \Omega_0 \Rightarrow v_I = 0 \Rightarrow \theta^0_{v,I} = 1.$$

and so

$$\{\theta_u^0 = 1\} \subset \{\theta_v^0 = 1\}.$$

Moreover, using (6.29), we deduce that for α small enough, we have, for all $t \ge 0$

$$c_{v,I} \ge (c(x_I,t))^+.$$

Using the comparison principle Corollary 4.5, we deduce that

$$\theta_v^\varepsilon(x,t) \ge \theta^\varepsilon(x,t).$$

We denote by $v^\varepsilon(x) = \sup_{y \in [x-\Delta x, x]} v(y)$ and $\theta_{v^\varepsilon}(x,t) = 1_{\{v^\varepsilon(x)\ge t\}} - 1_{\{v^\varepsilon(x)<t\}}$. It is easy to check that

$$(\theta_{v^\varepsilon})^*(x,t) \ge (\theta_v^\varepsilon)^*(x,t) \ge (\theta^\varepsilon)^*(x,t).$$

Passing to the limit $\varepsilon \to 0$, we then obtain for $t > 0$

$$1_{\{v(x)\ge t\}} - 1_{\{v(x)<t\}} = \theta_v(x,t) \ge \overline{\theta}^0(x,t)$$

and so

$$\left(1_{\Omega_0} - 1_{\Omega_0^c}\right)^* \ge \overline{\theta}^0(x,0).$$

This implies that $\overline{\theta}^0$ satisfies the initial condition (6.27). □

Proof of Theorem 2.5 The proof of Theorem 2.5 is now quite simple. Indeed, using Theorem 6.1 and Proposition 6.2, we get that $\overline{\theta}^0$ is a viscosity sub-solution of (1.1).

For the super-solution property of $\underline{\theta}^0$, it suffices to use the symmetry of $\overline{\theta}^0$ and $\underline{\theta}^0$ (see Lemma 4.1). Indeed, by contradiction, assume that there are (x_0,t_0) and $\varphi \in C^2$ such that $\underline{\theta}^0 - \varphi$ reaches a strict minimum at (x_0,t_0) with

$$\varphi_t(x_0,t_0) = -\alpha + c(x_0,t_0)|D\varphi(x_0,t_0)|$$

with $\alpha > 0$ and $t_0 > 0$. Let us define $c_1 = -c$, $\varphi_1 = -\varphi$ and $\overline{\theta}_1^0 = \overline{\theta}^0[-\theta^0, -c]$. Then, using Lemma 4.1, we get that $\overline{\theta}_1^0 - \varphi_1$ reaches a strict maximum at (x_0, t_0) with $\overline{\theta}_1^0(x_0, t_0) = \varphi_1(x_0, t_0)$ and

$$(\varphi_1)_t(x_0, t_0) = \alpha + c_1(x_0, t_0)|D\varphi(x_0, t_0)|.$$

This contradicts the sub-solution property of $\overline{\theta}_1^0$. For the initial condition, we use the same arguments of those of Proposition 6.2.

Moreover, if (1.1) satisfies a comparison principle, then $\overline{\theta}^0 \leq (\underline{\theta}^0)^*$ and $(\overline{\theta}^0)_* \leq \underline{\theta}^0$. Since, by definition, $\overline{\theta}^0 \geq \underline{\theta}^0$, we get that $\overline{\theta}^0 = (\underline{\theta}^0)^*$ and $(\overline{\theta}^0)_* = \underline{\theta}^0$ is a solution of (1.1). This exactly means that $\overline{\theta}^0$ and $\underline{\theta}^0$ are solutions, which is then unique (when the comparison principle holds for a special choice of the initial data), up to the upper and the lower semi-continuous envelopes. $\square$

7 Numerical tests

We are going to verify our algorithm by some numerical tests in dimension $N = 2$.
First we will give in two cases the representation formula of the solution so that we will be able to obtain numerical errors comparing it with the numerical solution obtained by the GFMM algorithm.

Representation formulas for hyperplanes and spheres propagating with linear speed
We verify that hyperplanes and spheres in $\mathbb{R}^N$, that propagate with a linear speed along the normal direction, keep their shapes during the evolution remaining respectively hyperplanes and spheres.
These manifolds can be characterized by the level set of a polynomial $P(x) : \mathbb{R}^N \to \mathbb{R}$ of degree 1 and 2. We denote by $P(x,t)$ the polynomials with coefficients depending on t.
Each point x s.t. $P(x, t_0) = 0$ verifies the following dynamics:

$$\begin{cases} \dot{y}(t) & = -c(y(t),t)\frac{DP(y(t),t)}{|DP(y(t),t)|}, \\ y(t_0) & = x \end{cases}$$

since they propagate with speed c along the unit normal to the manifold. These trajectories are known as *characteristics*. Then we just need to check that the evolution of each point of the manifold verifies the equation $P(y(t),t) = 0$, i.e. deriving with respect to t

$$P_t(y(t),t) - |DP(y(t),t)|c(y(t),t) = 0, \tag{7.30}$$

for any linear speed $c(x,t) = a(t)x + b(t)$ and for any $P(x,t)$ representing hyperplanes or spheres.

Hyperplanes: $P(x,t) = \alpha(t)x + \beta(t)$
It results $P_t(x,t) = \dot{\alpha}(t)x + \dot{\beta}(t)$ and $|DP(x(t),t)| = |\alpha(t)|$ then $P(x,t)$ verifies (7.30) with coefficients such that:

$$\begin{cases} \dot{\alpha}(t) = |\alpha(t)|a(t) \\ \dot{\beta}(t) = |\alpha(t)|b(t) \end{cases}$$

Spheres: $P(x,t) = R(t)^2 - |x - x_0(t)|^2$
It results $P_t(x,t) = 2(x - x_0(t))\dot{x}_0(t) + 2R(t)\dot{R}(t)$ and $|DP(x(t),t)| = 2|x - x_0(t)|$ then $P(x,t)$ verifies (7.30) with coefficients such that:

$$\begin{cases} \dot{x}_0(t) = a(t)R(t) \\ \dot{R}(t) = x_0(t)a(t) + b(t) \end{cases}$$

Test 1 : a rotating line
We choose as initial data a line $P(x,0) = x_2 + 1.5x_1$ and then as representing function:

$$\theta(x,0) = \begin{cases} 1 & \text{if } x_2 + 1.5x_1 > 0 \\ -1 & \text{otherwise.} \end{cases} \tag{7.31}$$

We choose as velocity $c(x,t) = x_1$. We have proved that a line propagating with linear speed stays a line. Applying the result of the previous section, we obtain that $P(x,t) = \alpha(t)x + \beta(t)$ has coefficients solving the following o.d.e.

$$\begin{cases} \dot{\alpha}_1(t) = \sqrt{1 + \alpha_1(t)^2} \\ \alpha_1(0) = 1.5, \end{cases} \qquad \begin{cases} \dot{\alpha}_2(t) = 0 \\ \alpha_2(0) = 1. \end{cases}$$

Solving, we obtain $P(x,t) = \sinh(t + \operatorname{arcsinh}(\alpha_1(0)))x_1 + x_2$.
We compute the discrete solution in the numerical domain $D = [-1,1] \times [-1,1]$ and we evaluate the error at final time T=0.5. We use the discrete L^1-norm

$$\|\theta(x_I,T) - \theta_I^m\|_1 = \sum_{\{I: x_I \in D\}} |\theta(x_I,T) - \theta_I^m| \Delta x^2,$$

with m the number of iterations corresponding to reach the final time T. The table 1 shows the error for the tests run with $26, 51, 101, 201$ number of nodes for each side of the square domain. The convergence is approximately of order 1.
Fig.15 shows the 0-level set of the discrete solution at each time interval 0.1. The line is rotating clockwise and it will reach in infinity time the x_2 axe. The last line is plotted with the exact solution in thicker line.

Δx	L^1-error
0.08	0.102
0.04	0.0576
0.02	0.0304
0.01	0.0160

Table 1: Numerical errors for test 1

Test 2 : propagation of a circle
We choose as initial data a circle $P(x,0) = x_1^2 + x_2^2 - 1$ and then as representing function:

$$\theta(x,0) = \begin{cases} 1 & x_1^2 + x_2^2 - 1 < 0 \\ -1 & \text{otherwise.} \end{cases} \tag{7.32}$$

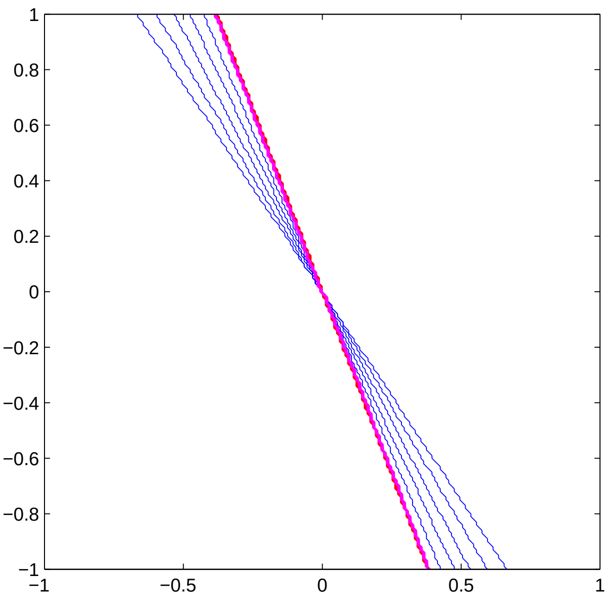

Figure 15: A rotating line

We choose as velocity $c(x,t) = 0.1t - x_1$. We have proved that a circle propagating with linear speed stays a circle. Applying the result of the previous section, we obtain that $P(x,t) = (x_1 - x_{0,1}(t))^2 + (x_2 - x_{0,2}(t))^2 - R(t)^2$ has coefficients solving the following o.d.e.

$$\begin{cases} \dot{x}_{0,1}(t) = -R(t) \\ x_{0,1}(0) = 0 \end{cases} \quad \begin{cases} \dot{x}_{0,2}(t) = 0 \\ x_{0,2}(0) = 0 \end{cases} \quad \begin{cases} \dot{R}(t) = -x_{0,1}(t) + 0.1t \\ R(0) = 1 \end{cases}$$

Solving, we obtain $x_{0,1}(t) = 1/20(2t + 11(\exp(-t) - \exp(t))$ and $R(t) = 1/20(-2 + 11(\exp(t) + \exp(-t)))$.
We compute the discrete solution in the numerical domain $D = [-2,2] \times [-2,2]$ and we evaluate the error at final time T=0.5.
We use the discrete L^1-norm, defined in the previous test.
The table 2 shows the error for the tests run with $51, 101, 201, 401$ number of nodes for each side of the square domain. The convergence is approximately of order 1.
Fig.16 shows the 0-level set of the discrete solution and at each time interval 0.1, the circle is expanding ad its centre is propagating on the left.

Δx	L^1-error
0.08	0.4992
0.04	0.2784
0.02	0.1288
0.01	0.0582

Table 2: Numerical errors for test 2

Test 3: comparison between the FMM and GFMM algorithm
When the evolution is monotone, i.e. $c(x) > 0$, there exists a link between the evolutive and

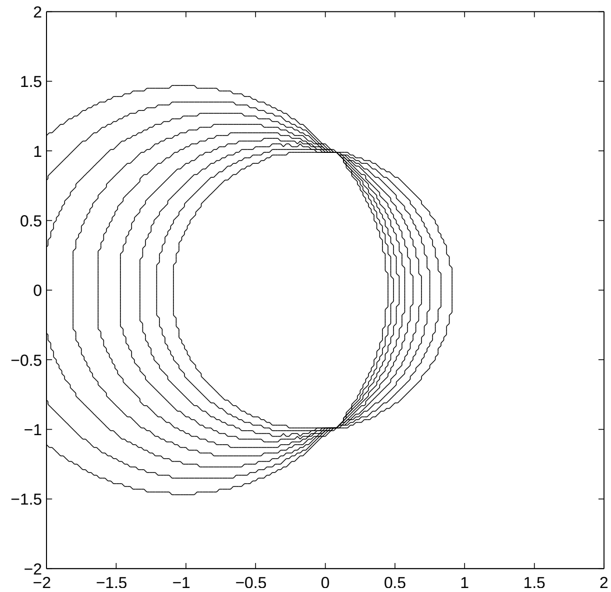

Figure 16: A propagating circle

the stationary equation(see [8] and [11]):

$$\begin{cases} c(x)|DT(x)| = 1 & x \in \Omega, \\ T(x) = 0 & x \in \partial\Omega. \end{cases}$$

In this case the discrete function u_I^n, computed by the GFMM algorithm, approximates the solution $T(x)$ outside the set Ω.
The two schemes, the FMM and the GFMM, are run in the case the speed is $c(x,t) = 1$ with initial set Ω a circle centred in the origin with radius 0.5.
For this choice of speed, the solution $T(x)$ corresponds at the distance function of the point x from the set Ω.
We compare the two schemes computing the errors in the $||\cdot||_\infty$ discrete norm:

$$\||T(x_I) - u_I\|_\infty \equiv \sup_{\{I: x_I \in D\}} |T(x_I) - u_I|.$$

As one can see, the GFMM scheme produces in this particular case almost the same results

Δx	FMM	GFMM
0.08	0.065	0.078
0.04	0.033	0.039
0.02	0.020	0.018

Table 3: Numerical errors for test 3

of the FMM scheme (as implemented in the HJpack library [22]). The results are slightly different in particular because the time computed in the narrow band in the classical FMM uses not only the accepted points but also the points of the narrow band.

Test 4: two collapsing circles

We choose as initial data two circles and as velocity $c(x,t) = 1 - t$. The two circles grow as

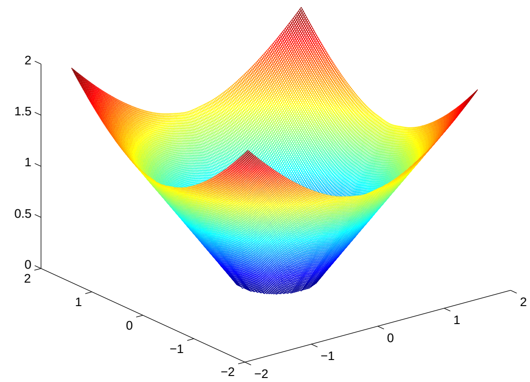

Figure 17: The discrete time u of a propagating circle with positive constant speed

far as the speed is positive. At $t = 1$, when the velocity changes sign, they start to decrease. Fig.18 on the left shows the 0-level set of the discrete solution at each time interval 0.2 until $t = 1$ and Fig.18 on the right shows the 0-level set of the discrete solution at each time interval 0.2 for the time interval $[1.2, 2.4]$.

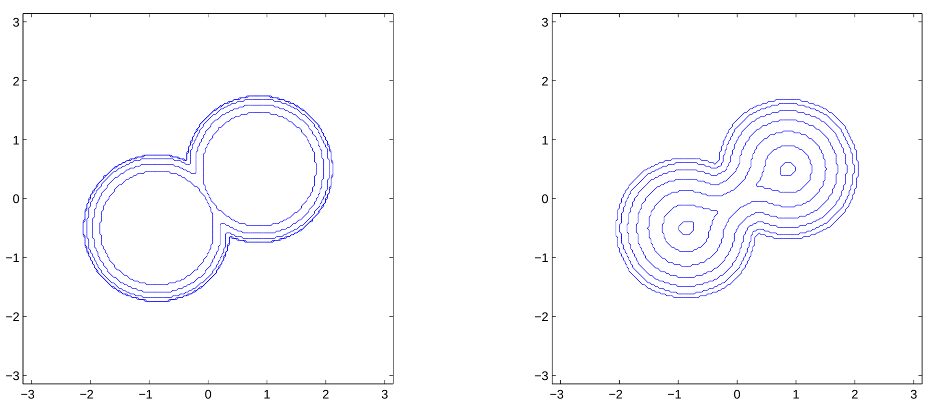

Figure 18: Two propagating circles

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